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ATOMICITY IN INJECTIVE BANACH LATTICES[1]

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To Semën Kutateladze
on occasion of his 70th birthday

This note is aimed to examine a Boolean valued interpretation of the concept of atomic Banach lattice and to give a complete description of the corresponding class of injective Banach lattices.

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1. Introduction

The aim of this note is to examine a Boolean valued interpretation of the concept of atomic Banach lattice and to give a complete description of the corresponding class of injective Banach lattices. Some representation and isometric classification results for general injective Banach lattices were announced in [1, 2].

Section 2 collects some needed Boolean valued representation results following [3]. In Section 3 we demonstrate that a Boolean valued interpretation of atomicity yields some "module atomicity" over a certain f-subalgebra of the center. Section 4 deals with Boolean valued Banach lattices of summable families, which turn out to be "building blocks" for general module atomic injective Banach lattices. Section 5 exposes the main results on representation and classification of injective Banach lattices with atomic Boolean valued representation, i. e. those which are atomic with respect to their natural f-module structure.

The needed information on the theory of Banach lattices can be found in [1, 5]. Recall some definitions and notation. A real Banach lattice X is said to be *injective* if, for every Banach lattice Y, every closed vector sublattice $Y_0 \subset Y$, and every positive linear operator $T_0 : Y_0 \to X$ there exists a positive linear extension $T : Y \to X$ of T_0 with $\|T_0\| = \|T\|$; see [5, Definition 3.2.3]. Equivalently, X is an injective Banach lattice if, whenever X is lattice isometrically imbedded into a Banach lattice Y, there exists a positive contractive projection from Y onto X; one more equivalence definition states that each positive operator from X to any Banach lattice admits a norm preserving positive extension to any Banach lattice containing X as a vector sublattice, see [3, Theorem 5.10.6]. This concept was introduced by Lotz [6]; a significant advance towards the structure theory of injectives was made by Cartwright [7] and Haydon [8].

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In what follows X stands for a real Banach lattice. We denote by $\mathbb{P}(X)$ the Boolean algebra of all band projections in X. A crucial role in the theory of injective Banach lattices is played by the concept of M-projection. A band projection π in a Banach lattice X is called an *M-projection* if $\|x\| = \max\{\|\pi x\|, \|\pi^{\perp} x\|\}$ for all $x \in X$, where $\pi^{\perp} := I_X - \pi$. The collection $\mathbb{M}(X)$ of all M-projections in X is a subalgebra of the Boolean algebra $\mathbb{P}(X)$.

Throughout the sequel $\mathbb{B}$ is a complete Boolean algebra with unit $\mathbb{1}$ and zero $\mathbb{O}$, while $\Lambda := \Lambda(\mathbb{B})$ is a Dedekind complete unital AM-space such that $\mathbb{B}$ is isomorphic to $\mathbb{P}(\Lambda)$. The unit of Λ is also denoted by $\mathbb{1}$. A *partition of unity* in $\mathbb{B}$ is a family $(b_\xi)_{\xi \in \Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi \in \Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{O}$ whenever $\xi \neq \eta$. We let $:=$ denote the assignment by definition, while $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ symbolize the naturals, the rationals, and the reals.

2. Boolean Valued Representation

Boolean valued analysis is an useful tool in studying of injective Banach lattices [9]. We need some Boolean valued representation results as presented in [3] and [25].

Applying the Transfer and Maximum Principles to the ZFC-theorem "There exists a field of reals" we find an element $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![\mathscr{R} \text{ is a field of reals}]\!] = \mathbb{1}$. We call $\mathscr{R}$ the *reals* within $\mathbb{V}^{(\mathbb{B})}$. The following remarkable result due to Gordon [28] tells us that the interpretation of the reals in $\mathbb{V}^{(\mathbb{B})}$ is a universally complete vector lattice with the Boolean algebra of band projections isomorphic to $\mathbb{B}$.

Theorem 2.1. *Let $\mathscr{R}$ be the reals within $\mathbb{V}^{(\mathbb{B})}$. Then $\mathscr{R}{\downarrow}$ (with the descended operations and order) is a universally complete vector lattice with a weak order unit $\mathbb{1} := 1^{\wedge}$. Moreover, there exists a Boolean isomorphism χ of $\mathbb{B}$ onto $\mathbb{P}(\mathscr{R}{\downarrow})$ such that the equivalences*

$$\begin{aligned} \chi(b)x = \chi(b)y &\iff b \leqslant [\![x = y]\!], \\ \chi(b)x \leqslant \chi(b)y &\iff b \leqslant [\![x \leqslant y]\!] \end{aligned} \tag{$\mathbb{G}$}$$

hold for all $x, y \in \mathscr{R}{\downarrow}$ and $b \in \mathbb{B}$.

$\lhd$ See [3, Theorem 2.2.4] and [25, Theorem 10.3.4]. $\rhd$

Definition 2.2. A *complete Boolean algebra of M-projections* in X is an arbitrary order complete and order closed subalgebra $\mathbb{B} \subset \mathbb{M}(X)$. A Banach lattice X is said to be $\mathbb{B}$-*cyclic* whenever it is a $\mathbb{B}$-cyclic Banach space with respect to a complete Boolean algebra $\mathbb{B}$ of M-projections. If X has the Fatou and Levi properties (see [3, 5.7.2]), then $\mathbb{M}(X)$ itself is an order closed subalgebra of the complete Boolean algebra $\mathbb{P}(X)$.

Definition 2.3. Let $\Lambda = \mathscr{R}{\Downarrow}$ be the bounded part of the universally complete vector lattice $\mathscr{R}{\downarrow}$; i. e., Λ is the order-dense ideal in $\mathscr{R}{\downarrow}$ generated by the weak order unit $\mathbb{1} := 1^{\wedge} \in \mathscr{R}{\downarrow}$. Take a Banach space $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$ and put $\mathscr{X}{\Downarrow} := \{x \in \mathscr{X}{\downarrow} : |\!x\!| \in \Lambda\}$. Equip $\mathscr{X}{\Downarrow}$ with some *mixed norm* by putting $\|x\| := \||\!x\!|\|_\infty$ for all $x \in X$, where the order unit norm $\|\cdot\|_\infty$ is defined as $\|\lambda\|_\infty := \inf\{0 < \alpha \in \mathbb{R} : |\lambda| \leqslant \alpha \mathbb{1}\}$ $(\lambda \in \Lambda)$. In this situation, $(\mathscr{X}{\Downarrow}, \|\cdot\|)$ is a Banach space called the *bounded descent* of $\mathscr{X}$. The terms $\mathbb{B}$-isomorphism and $\mathbb{B}$-isometry mean that isomorphism or isometry under consideration commutes with the projections from $\mathbb{B}$, see [3, 5.8.9].

Theorem 2.4. *A bounded descent of a Banach lattice from the model $\mathbb{V}^{(\mathbb{B})}$ is a $\mathbb{B}$-cyclic Banach lattice. Conversely, if X is a $\mathbb{B}$-cyclic Banach lattice, then in the model $\mathbb{V}^{(\mathbb{B})}$ there exists up to the isometric isomorphism a unique Banach lattice $\mathscr{X}$ whose bounded descent is isometrically $\mathbb{B}$-isomorphic to X. Moreover, $\mathbb{B} = \mathbb{M}(X)$ if and only if $[\![$there is no M-projection in $\mathscr{X}$ other than 0 and $\mathscr{X}]\!] = \mathbb{1}$.*

$\lhd$ See [3, Theorem 5.9.1]. $\rhd$

Definition 2.5. The element $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ from Theorem 2.1 is said to be the *Boolean-valued representation of* X.

Theorem 2.6. *Let X be a Banach lattice with the complete Boolean algebra $\mathbb{B} = \mathbb{M}(X)$ of M-projections, Λ be a Dedekind complete unital AM-space such that $\mathbb{P}(\Lambda)$ is isomorphic to $\mathbb{B}$. Then the following assertions are equivalent:*

(1) *X is injective.*

(2) *X is lattice $\mathbb{B}$-isometric to the bounded descent of some AL-space from $\mathbb{V}^{(\mathbb{B})}$.*

(3) *There exists a strictly positive Maharam operator $\Phi : X \to \Lambda$ with the Levi property such that $X = L^1(\Phi)$ and $\|x\| = \|\Phi(|x|)\|_\infty$ for all $x \in X$.*

(4) *There is a Λ-valued additive norm on X such that $(X, |\!\cdot\!|)$ is a Banach–Kantorovich lattice and $\|x\| = \big\| |x| \big\|_\infty$ for all $x \in X$.*

$\lhd$ See [3, Theorem 5.12.5]. $\rhd$

Theorem 2.7. *Suppose that X is a Banach lattice and $\mathscr{X}$ is the completion of the metric space $X^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. Then $[\![\mathscr{X}$ is a Banach lattice$]\!] = \mathbb{1}$ and $\mathscr{X}{\Downarrow}$ is lattice $\mathbb{B}$-isometric to $C_\#(Q, X)$ equipped with the norm $\|\varphi\| = \sup\{\|\varphi(q)\| : q \in \mathrm{dom}(\varphi) \subset Q\}$ $(\varphi \in C_\#(Q, X))$.*

$\lhd$ The proof is a due modification of [25, 11.3.8]. $\rhd$

3. Boolean Valued Atomicity

In this section we present Boolean valued interpretation of atomicity.

Definition 3.1. A positive element x of a $\mathbb{B}$-cyclic Banach lattice X is said to be $\mathbb{B}$-*indecomposable* or a $\mathbb{B}$-*atom* if for any pair of disjoint elements $y, z \in X_+$ with $y + z \leqslant x$ there exists a projection $\pi \in \mathbb{B}$ such that $\pi y = 0$ and $\pi^\perp z = 0$, while X is called $\mathbb{B}$-*atomic* if the only element of X disjoint from every $\mathbb{B}$-atom is the zero element.

Denote by $\mathrm{at}(\mathscr{X})$ and $\mathbb{B}$-$\mathrm{at}(X)$ the sets of atoms in $\mathscr{X}$ and $\mathbb{B}$-atoms in X, respectively. Let $\mathrm{at}_1(\mathscr{X}) := \{x \in \mathrm{at}(\mathscr{X}) : \|x\| = 1\}$, while $\mathbb{B}$-$\mathrm{at}_1(X)$ consists of all $x \in \mathbb{B}$-$\mathrm{at}(X)$ with $\|\pi x\| = 1$ for all $\pi \in \mathbb{B}$. It is easy to see that $\mathbb{B}$-$\mathrm{at}_1(X) = \{x \in \mathbb{B}\text{-}\mathrm{at}(X) : |x| = \mathbb{1}\}$.

Proposition 3.2. *Let X be a $\mathbb{B}$-cyclic Banach lattice identified with the bounded descent $\mathscr{X}{\Downarrow}$ of a Banach lattice $\mathscr{X}$, its Boolean valued representation $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$. Then the following assertions hold:*

(1) $\mathbb{B}$-$\mathrm{at}(X) = \mathrm{at}(\mathscr{X}){\Downarrow}$.

(2) $\mathbb{B}$-$\mathrm{at}_1(X) = \mathrm{at}_1(\mathscr{X}){\Downarrow}$.

(3) *X is $\mathbb{B}$-atomic if and only if $[\![\mathscr{X}$ is atomic$]\!] = \mathbb{1}$.*

$\lhd$ (1) Observe that $x \in \mathrm{at}(\mathscr{X})$ if and only if $x \in \mathscr{X}_+$ and for any two positive disjoint elements $x_1, x_2 \in \mathscr{X}$ with $x_1 + x_2 \leqslant x$ we have $x_1 = 0$ or $x_2 = 0$. Now, given $x \in \mathrm{at}(\mathscr{X}){\Downarrow}$ with $y + z \leqslant x$ for some disjoint $y, z \in X_+$, we put $b := [\![y = 0]\!]$ and $\pi := \chi(b)$. Since $[\![y \neq 0 \to z = 0]\!] = \mathbb{1}$, we have $[\![y \neq 0]\!] \leqslant [\![z = 0]\!]$ and thus $b^* = [\![y \neq 0]\!] \leqslant [\![z = 0]\!]$. By ($\mathbb{G}$) we have $\pi y = 0$ and $\pi^\perp z = \chi(b^*) z = 0$. Thus, $\mathrm{at}(\mathscr{X}){\Downarrow} \subset \mathbb{B}$-$\mathrm{at}(X)$ and for the converse inclusion the argument is similar.

(2) Taking into account the representation $\mathbb{B}$-$\mathrm{at}_1(X) = \{x \in \mathbb{B}\text{-}\mathrm{at}(X) : |x| = \mathbb{1}\}$ the claim follows easily from the following chain of equivalences:

$$
\begin{aligned}
x \in \mathrm{at}_1(\mathscr{X}){\Downarrow} \iff [\![x \in \mathrm{at}_1(\mathscr{X})]\!] = \mathbb{1} &\iff [\![x \in \mathrm{at}(\mathscr{X})]\!] = [\![\|x\|_{\mathscr{X}} = 1]\!] = \mathbb{1} \\
&\iff x \in \mathbb{B}\text{-}\,\mathrm{at}(\mathscr{X}) \ \wedge\ |x| = \mathbb{1} \iff x \in \mathbb{B}\text{-}\,\mathrm{at}_1(X).
\end{aligned}
$$

(3) Let for a while $\perp$, $\perp\!\!\!\perp$, and $\perp\!\!\!\perp\!\!\!\perp$ stand for disjoint complements in $\mathscr{X}$, $X = \mathscr{X}{\Downarrow}$, and $\mathscr{X}{\downarrow}$, respectively. The third claim is immediate from the first one, since the disjoint complement and the descent commute: $(A^{\perp}){\downarrow} = (A{\downarrow})^{\perp\!\!\!\perp\!\!\!\perp}$, see [3, 1.5.3]. Indeed,

$$(A^{\perp}){\Downarrow} = (A^{\perp}){\downarrow} \cap X = (A{\downarrow})^{\perp\!\!\!\perp\!\!\!\perp} \cap X = (A{\downarrow} \cap X)^{\perp\!\!\!\perp\!\!\!\perp} \cap X = (A{\Downarrow})^{\perp\!\!\!\perp},$$

hence putting $A := \mathrm{at}(\mathscr{X})$ and making use of (1) we deduce that $\mathrm{at}(\mathscr{X})^{\perp} = \{0\}$ within $\mathbb{V}^{(\mathbb{B})}$ if and only if $(\mathbb{B}\text{-at}(X))^{\perp\!\!\!\perp} = \{0\}$. $\triangleright$

Corollary 3.3. *Let* $\mathbb{B}$, X, *and* $\mathscr{X}$ *be the same as in Proposition* 3.2 *and* $\Lambda = \Lambda(\mathbb{B})$. *Then the following assertions hold:*

(1) $x \in X_{+}$ *is a* $\mathbb{B}$*-atom if and only if for each* $0 \leqslant y \leqslant x$ *there exists* $\lambda \in \Lambda_{+}$ *with* $y = \lambda x$.

(2) *If* x *and* y *are* $\mathbb{B}$*-atoms in* X_{+} *then there exist a pair of disjoint projections* $\pi, \rho \in \mathbb{B}$ *such that* $\pi x \perp \pi y$, $\rho x = \lambda u$ *and* $\rho y = \mu u$ *for some* $\mu, \lambda \in \Lambda_{+}$ *and* $u = x + y$.

$\triangleleft$ Interpreting in the model $\mathbb{V}^{(\mathbb{B})}$ the well-known claims corresponding to that particular case when $\mathbb{B} = \{0, I_X\}$ (see [13, Theorem 26.4.]) and using Proposition 3.2 yields the required properties. $\triangleright$

Definition 3.4. Given a cardinal γ, say that a $\mathbb{B}$-cyclic Banach lattice X is *purely* $(\mathbb{B}, \gamma)$*-atomic* if $X = \mathscr{D}_0^{\perp\perp}$ for some subset $\mathscr{D}_0 \subset \mathbb{B}\text{-at}_1(X)$ of cardinality γ and for every nonzero projection $\pi \in \mathbb{B}$ and every subset $\mathscr{D} \subset \mathbb{B}\text{-at}_1(\pi X)$ with $\pi X = \mathscr{D}^{\perp\perp}$ we have $\mathrm{card}(\mathscr{D}) \geqslant \gamma$. Evidently, X is purely $(\{0, I_X\}, \gamma)$-atomic if and only if X is atomic and the cardinality of $\mathrm{at}_1(X)$ is γ or, equivalently, X is atomic and the cardinality of the set of atoms in $\mathbb{B}(X)$ equals γ. In this case we say also that X is γ*-atomic.*

Proposition 3.5. *A* $\mathbb{B}$*-cyclic Banach lattice* X *is purely* $(\mathbb{B}, \gamma)$*-atomic for some cardinal* γ *if and only if* $[\![\, \gamma^{\wedge}$ *is a cardinal and* $\mathscr{X}$ *is* $\gamma^{\wedge}$*-atomic* $]\!] = \mathbb{1}$.

$\triangleleft$ *Sufficiency.* Assume that $\gamma^{\wedge}$ is a cardinal and $\mathscr{X}$ is $\gamma^{\wedge}$-atomic within $\mathbb{V}^{(\mathbb{B})}$. The latter means that $\mathscr{X}$ is atomic and $\mathrm{card}(\mathrm{at}_1(\mathscr{X})) = \gamma^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$. If $\Delta := \mathrm{at}_1(\mathscr{X})$ then there exists $\phi \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\phi : \gamma^{\wedge} \to \Delta$ is a bijection$]\!] = \mathbb{1}$. Note that $\phi{\downarrow}$ embeds γ into $\Delta{\downarrow}$ by [3, 1.5.8] and $\Delta{\downarrow} = \mathbb{B}\text{-at}_1(X)$ by Proposition 3.1. It follows that the set $\mathscr{D} := \phi{\downarrow}(\gamma)$ of cardinality γ is contained in $\mathbb{B}\text{-at}_1(X)$ and $X = \mathscr{D}^{\perp\perp}$, since $\Delta = \mathscr{D}{\uparrow}$ and $\mathscr{X} = \Delta^{\perp\perp}$. Take $b \in \mathbb{B}$ and a set $\mathscr{D}'$ of cardinality β which is contained in $\mathbb{B}\text{-at}_1(X)$ and generates bX, i. e. $bX = (\mathscr{D}')^{\perp\perp}$. Then $\mathscr{D}'{\uparrow}$ is of cardinality $\mathrm{card}(\beta^{\wedge})$ and $\mathscr{X} = (\mathscr{D}'{\uparrow})^{\perp\perp}$ within the relative universe $\mathbb{V}^{([\mathbb{O},b])}$. By [3, 1.3.7] $[\![\gamma^{\wedge} = \mathrm{card}(\gamma^{\wedge}) \leqslant \mathrm{card}(\beta^{\wedge}) \leqslant \beta^{\wedge}]\!] = \mathbb{1}$ and so $\gamma \leqslant \beta$.

Necessity. Assume now that X is purely $(\mathbb{B}, \gamma)$-atomic and $X = \mathscr{D}^{\perp\perp}$ for some $\mathscr{D} \subset \mathbb{B}\text{-at}_1(X)$ of cardinality γ. Then within $\mathbb{V}^{(\mathbb{B})}$ we have $\Delta := \mathscr{D}{\uparrow} \subset \mathrm{at}_1(\mathscr{X})$, $\mathscr{X} = \Delta^{\perp\perp}$ and and the cardinalities of Δ and $\gamma^{\wedge}$ coincide, i. e. $\mathrm{card}(\Delta) = \mathrm{card}(\gamma^{\wedge})$. By [3, 1.9.11] the cardinal $\mathrm{card}(\gamma^{\wedge})$ has the representation $\mathrm{card}(\gamma^{\wedge}) = \mathrm{mix}_{\alpha \leqslant \gamma} b_{\alpha} \alpha^{\wedge}$, where $(b_{\alpha})_{\alpha \leqslant \gamma}$ is a partition of unity in $\mathbb{B}$. It follows that $b_{\alpha} \leqslant [\![\Delta^{\perp\perp} = X$ and Δ is of cardinality $\alpha^{\wedge}]\!] = \mathbb{1}$. If $b_{\alpha} \neq \mathbb{O}$ then $(b_{\alpha} \wedge \Delta)^{\perp\perp} = b_{\alpha} \wedge \mathscr{X}$ and $b_{\alpha} \wedge \Delta$ is of cardinality $\mathrm{card}(\gamma^{\wedge}) = \alpha^{\wedge} \leqslant \gamma^{\wedge}$ in the relative universe $\mathbb{V}^{[\mathbb{O}, b_{\alpha}]}$. (Concerning $b_{\alpha} \wedge \Delta$ and $b_{\alpha} \wedge \mathscr{X}$ and their properties see [3, 1.3.7].) It is easy that $b_{\alpha} \wedge \Delta = (b_{\alpha}\mathscr{D}){\uparrow}$ and so $(b_{\alpha}\mathscr{D})^{\perp\perp} = bX$. By hypothesis X is purely $(\mathbb{B}, \gamma)$-atomic, consequently, $\alpha \geqslant \mathrm{card}(b_{\alpha}\mathscr{D}) \geqslant \gamma$, so that $\alpha = \gamma$, since $\alpha \leqslant \gamma$ if and only if $\alpha^{\wedge} \leqslant \gamma^{\wedge}$. Thus, $\mathrm{card}(\gamma^{\wedge}) = \gamma^{\wedge}$ whenever $b_{\alpha} \neq \mathbb{O}$ and $\gamma^{\wedge}$ is a cardinal within $\mathbb{V}^{(\mathbb{B})}$. $\triangleright$

Definition 3.6. Let γ is a cardinal. A complete Boolean algebra $\mathbb{B}$ (as well as its Stone representation space) is said to be γ*-stable* whenever $\mathbb{V}^{(\mathbb{B})} \models \gamma^{\wedge} = \mathrm{card}(\gamma^{\wedge})$, i. e. $[\![\gamma^{\wedge}$ is a cardinal $]\!] = \mathbb{1}$. An element $b \in \mathbb{B}$ is called γ*-stable* if the relative Boolean algebra $[\mathbb{O}, b]$ is γ-stable, see [25, Definition 12.3.7]. Finally, say that a partition of unity $(\pi_{\gamma})_{\gamma \in \Gamma}$ in $\mathbb{B}$ with Γ a set of cardinals is *stable* if π_{γ} is γ-stable for all $\gamma \in \Gamma$.

Theorem 3.7. *Let X be a $\mathbb{B}$-atomic $\mathbb{B}$-cyclic Banach lattice. There exist a set of cardinals Γ and a partition of unity $(\pi_\gamma)_{\gamma\in\Gamma}$ such that $\mathbb{B}_\gamma := [\mathbb{O}, \pi_\gamma]$ is γ-stable and $\pi_\gamma X$ is purely $(\mathbb{B}_\gamma, \gamma)$-atomic for all $\gamma \in \Gamma$.*

$\lhd$ If a $\mathbb{B}$-cyclic Banach lattice X is $\mathbb{B}$-atomic then its Boolean valued representation $\mathscr{X}$ is atomic within $\mathbb{V}^{(\mathbb{B})}$ according to Proposition 3.1. Denote $\gamma_0 := \operatorname{card}(\operatorname{at}_1(\mathscr{X}))$. By [3, 1.9.11] γ_0 is a mixture of some set of relatively standard cardinals. More precisely, there are nonempty set of cardinals Γ and a partition of unity $(b_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{B}$ such that $x = \operatorname{mix}_{\gamma\in\Gamma} b_\gamma\gamma^\wedge$ and $\mathbb{V}^{(\mathbb{B}_\gamma)} \models \gamma^\wedge = \operatorname{card}(\gamma^\wedge)$ with $\mathbb{B}_\gamma := [\mathbb{O}, b_\gamma]$ for all $\gamma \in \Gamma$. It follows that $b_\gamma \wedge \mathscr{X}$ is atomic Banach lattice and $\gamma^\wedge = \operatorname{card}(\operatorname{at}_1(b_\gamma \wedge \mathscr{X}))$ within $\mathbb{V}^{(\mathbb{B}_\gamma)}$. It remains to apply Proposition 3.5. $\rhd$

4. The Banach Lattices $l^1(\Gamma, \Lambda)$ and $C_\#(Q, l^1(\Gamma))$

We now consider some special injective Banach lattices that are building blocks for the class of all $\mathbb{B}$-atomic injective Banach lattices. Recall that $\Lambda = \Lambda(\mathbb{B})$.

Given a non-empty set Γ, denote by $l^1(\Gamma^\wedge) \in \mathbb{V}^{(\mathbb{B})}$ the internal Banach lattice of all summable families $x := (x_\gamma)_{\gamma\in\Gamma^\wedge}$ in $\mathscr{R}$ with the norm $\|x\|_1 := \sum_{\gamma\in\Gamma^\wedge} |x_\gamma|$.

Let $l^1(\Gamma, \Lambda)$ stand for the vector space of all order summable families in Λ, i.e.

$$l^1(\Gamma,\Lambda) := \Big\{\mathbf{x} : \Gamma \to \Lambda : \ \boldsymbol{|}\mathbf{x}\boldsymbol{|}_1 := o\text{-}\sum\nolimits_{\gamma\in\Gamma} |\mathbf{x}(\gamma)| \in \Lambda\Big\}.$$

The order on $l^1(\Gamma, \Lambda)$ is defined by letting $\mathbf{x} \leqslant \mathbf{y}$ if and only if $\mathbf{x}(\gamma) \leqslant \mathbf{y}(\gamma)$ for all $\gamma \in \Gamma$. Evidently, $l^1(\Gamma, \Lambda)$ is an order ideal of the Dedekind complete vector lattice Λ^Γ, hence so is $l^1(\Gamma, \Lambda)$. Moreover, $l^1(\Gamma, \Lambda)$ equipped with the norm $\|\mathbf{x}\| := \|\boldsymbol{|}\mathbf{x}\boldsymbol{|}_1\|_\infty$ $(\mathbf{x} \in l^1(\Gamma, \Lambda))$ is a $\mathbb{B}$-cyclic Banach lattice, since $\mathbb{B} = \mathbb{B}(\Lambda)$.

Proposition 4.1. *$l^1(\Gamma^\wedge)$ is a Boolean valued representation of $l^1(\Gamma, \Lambda)$ and thus $l^1(\Gamma, \Lambda)$ and $l^1(\Gamma^\wedge){\Downarrow}$ are lattice $\mathbb{B}$-isometric.*

$\lhd$ Straightforward verification shows that $l^1(\Gamma, \Lambda)$ is a Banach f-module over Λ, see [3, Definitions 2.11.1 and 5.7.1]. The modified ascent mapping $\mathbf{x} \mapsto \mathbf{x}{\Uparrow}$ is a bijection from $(\mathscr{R}{\downarrow})^\Gamma$ onto $(\mathscr{R}^{\Gamma^\wedge}){\downarrow}$, see [3, 1.5.9]. It follows from [3, 2.4.7] that $\boldsymbol{|}\cdot\boldsymbol{|}_1$ is the bounded descent of $\|\cdot\|_1$ and hence $\mathbf{x} \in l^1(\Gamma, \Lambda)$ if and only if $[\![\mathbf{x}{\Uparrow} \in l^1(\Gamma^\wedge)]\!] = \mathbb{1}$. Moreover, in this event $[\![\boldsymbol{|}\mathbf{x}\boldsymbol{|}_1 = \|\mathbf{x}{\Uparrow}\|_1]\!] = \mathbb{1}$ so that the modified descent induces an isometric bijection between $l^1(\Gamma, \Lambda)$ and $(l^1\Gamma^\wedge){\Downarrow}$. Making use of the definition of modified descent it can be easily checked that this bijection is Λ-linear and order preserving. $\rhd$

Proposition 4.2. *The Banach lattice $l^1(\Gamma, \Lambda)$ is $\mathbb{B}$-atomic and injective with $\mathbb{M}(X)$ isomorphic to $\mathbb{B}$. Moreover, $l^1(\Gamma, \Lambda)$ is purely $(\mathbb{B}, \gamma)$-atomic if and only if $[\![\gamma^\wedge = \operatorname{card}(\Gamma^\wedge)]\!] = \mathbb{1}$.*

$\lhd$ By Theorem 2.6 (2) and Propositions 3.2 and 4.1 X is injective with $\mathbb{M}(X) \simeq \mathbb{B}$ and $\mathbb{B}$-atomic. The second part follows from Propositions 3.5 and 4.1, since $l^1(\Gamma^\wedge)$ is $\operatorname{card}(\Gamma^\wedge)$-atomic within $\mathbb{V}^{(\mathbb{B})}$. $\rhd$

Proposition 4.3. *The norm completion of $\mathbb{R}^\wedge$-normed space $l^1(\Gamma)^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$ is a Banach lattice which is lattice isometric to the internal Banach lattice $l^1(\Gamma^\wedge)$.*

$\lhd$ Denote by $\mathscr{L}_1$ the completion of $l^1(\Gamma)^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$. Let A be the set of all norm-one atoms in $l^1(\Gamma)$ which is of course bijective with Γ. Then $A^\wedge$ and $\Gamma^\wedge$ are also bijective and $A^\wedge$ can be considered as the set of all norm-one atoms in $l^1(\Gamma^\wedge)$. Denote by $\mathbb{Q}$-lin(A) the set of all linear combinations of the members of A with rational coefficients. Then by [12, 8.4.10] we have $(\mathbb{Q}\text{-lin}(A))^\wedge = \mathbb{Q}^\wedge\text{-lin}(A^\wedge)$. Clearly, $\mathbb{Q}^\wedge$-lin$(A^\wedge)$ is a dense sublattice in $l^1(\Gamma^\wedge)$, while $(\mathbb{Q}\text{-lin}(A))^\wedge$ is a dense sublattice in $l^1(\Gamma)^\wedge$ and thus in $\mathscr{L}_1$, since $\mathbb{Q}$-lin(A) is dense in $l^1(\Gamma)$. Moreover, the norms induced in $(\mathbb{Q}\text{-lin}(A))^\wedge$ by $l^1(\Gamma^\wedge)$ and $l^1(\Gamma)^\wedge$ coincide. Indeed, if $x \in (\mathbb{Q}\text{-lin}(A))^\wedge$ is of the form $\sum_{k\in n} r(k)\,u(k)$ whith $n \in \mathbb{N}$, $r : n \to \mathbb{Q}$, and $u : n \to A$, then $r^\wedge : n^\wedge \to \mathbb{Q}^\wedge$, $u^\wedge : n^\wedge \to A^\wedge$ and $x^\wedge = \sum_{k\in n^\wedge} r^\wedge(k)u^\wedge(k)$; therefore,

$$\|x\|_{l^1(\Gamma)^\wedge} = \|x\|^\wedge = \Big(\sum\nolimits_{k\in n} |r(k)|\Big)^\wedge = \sum\nolimits_{k\in n^\wedge} |r^\wedge(k)| = \|x\|_{l^1(\Gamma^\wedge)}.$$

It follows that $\mathscr{L}_1$ and $l^1(\Gamma^\wedge)$ are lattice isometric. $\rhd$

Corollary 4.4. *Let Q be the Stone representation space of $\mathbb{B} = \mathbb{P}(\Lambda)$. Then the injective Banach lattices $l^1(\Gamma, \Lambda)$ and $C_\#(Q, l^1(\Gamma))$ are lattice $\mathbb{B}$-isometric.*

$\lhd$ This is immediate from Theorem 2.7 and Proposition 4.3. $\rhd$

Corollary 4.5. *Given an arbitrary infinite cardinals γ_1 and γ_2, we may find a Boolean algebra $\mathbb{B}$ such that the injective Banach lattices $l^1(\gamma_1, \Lambda)$ and $l^1(\gamma_2, \Lambda)$ are lattice $\mathbb{B}$-isometric provided that $\Lambda = \Lambda(\mathbb{B})$. If Q is the Stone representation space of $\mathbb{B}$ then the injective Banach lattices $C_\#(Q,\ l^1(\gamma_1)$ and $C_\#(Q, l^1(\gamma_2))$ are also lattice $\mathbb{B}$-isometric.*

$\lhd$ The claim follows from Proposition 4.3 and Corollary 4.4 making use of the *cardinal collapsing* phenomena: There exists a complete Boolean algebra $\mathbb{B}$ such that the ordinals $\gamma_1^\wedge$ and $\gamma_2^\wedge$ have the same cardinality within $\mathbb{V}^{(\mathbb{B})}$, see [3, 1.13.9]. $\rhd$

Definition 4.6. A $\mathbb{B}$-cyclic Banach lattice X is called $\mathbb{B}$-*separable,* if there is a sequence $(x_n) \subset X$ such that the norm closed $\mathbb{B}$-cyclic subspace, generated by the set $\{bx_n : n \in \mathbb{N},\ b \in \mathbb{B}\}$, coincides with X. In more detail, X is called $\mathbb{B}$-separable whenever for every $x \in X$ and $0 < \varepsilon \in \mathbb{R}$ there exist an element $x_\varepsilon \in X$ and a partition of unity $(\pi_n)_{n\in\mathbb{N}}$ in $\mathbb{B}$ such that $\|x - x_\varepsilon\| \leqslant \varepsilon$ and $\pi_n x = \pi_n x_n$ for all $n \in \mathbb{N}$. It can be easily seen that X is $\mathbb{B}$-separable if and only if its Boolean valued representation is separable within $\mathbb{V}^{(\mathbb{B})}$. Denote by ω the countable cardinal and put $l^1 := l^1(\omega)$.

Corollary 4.7. *For every infinite cardinal γ, there exists a Stonean space Q such that the injective Banach lattice $C_\#(Q, l^1(\gamma))$ is $\mathbb{B}$-separable, with $\mathbb{B}$ standing for the Boolean algebra of the characteristic functions of clopen subsets of Q.*

$\lhd$ Apply Corollary 4.5 with $\gamma_1 := \gamma$ and $\gamma_2 := \omega$, where ω is the countable cardinal. It follows that $C_\#(Q,\ l^1(\gamma)$ and $C_\#(Q, l^1(\omega))$ are lattice $\mathbb{B}$-isometric. Moreover, $[\![l^1(\omega^\wedge)$ is separable$]\!] = \mathbb{1}$ by transfer principle. Taking into account Proposition 4.1 it remains to observe that $[\![\mathscr{X}$ is separable$]\!] = \mathbb{1}$ if and only if $\mathscr{X}{\Downarrow}$ is $\mathbb{B}$-separable. $\rhd$

5. The Main Results

Now we are able to state and prove the main representation and classification results for $\mathbb{B}$-atomic injective Banach spaces.

Definition 5.1. Let X be an injective Banach lattice. Say that X is *centrally atomic* if X is $\mathbb{B}$-atomic with $\mathbb{B} = \mathbb{M}(X)$. According to corollary 3.3 this amounts to saying that there is no nonzero element in X disjoint from all Λ-atom, while a Λ-*atom* is any element $x \in X_+$ such that the principal ideal generated by x is equal to $\Lambda x := \{\lambda x : \lambda \in \Lambda\}$. Given a family of Banach lattices $(X_\gamma, \|\cdot\|_\gamma)_{\gamma\in\Gamma}$, denote by $\big(\sum^\oplus_{\gamma\in\Gamma} b_\gamma X\big)_{l^\infty}$ the l^∞-*sum,* the Banach lattice of all families $\mathbf{x} := \big(\mathbf{x}(\gamma)\big)_{\gamma\in\Gamma}$ with $\mathbf{x}(\gamma) \in X_\gamma$ for all $\gamma \in \Gamma$ and $\|\mathbf{x}\| := \sup\{\|\mathbf{x}(\gamma)\|_\gamma : \gamma \in \Gamma\} < \infty$.

Lemma 5.2. *For a centrally atomic injective Banach lattice X there exist a set of cardinals Γ and a stable partition of unity $(\pi_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{M}(X)$ such that $\pi_\gamma X$ is purely $(\gamma, \mathbb{B}_\gamma)$-atomic with $\mathbb{B}_\gamma := [\mathbb{O}, \pi_\gamma]$ for all $\gamma \in \Gamma$ and injective and the representation holds:*

$$X \simeq_{\mathbb{B}} \Big(\sum\nolimits^\oplus_{\gamma\in\Gamma} b_\gamma X\Big)_{l^\infty}.$$

◁ This is immediate from Proposition 3.7. ▷

Lemma 5.3. *Suppose that the injective Banach lattices $C_\#(Q, l^1(\gamma))$ and $C_\#(Q, l^1(\delta))$ are lattice $\mathbb{B}$-isometric, where Q is the Stone space of $\mathbb{B}$, while γ and δ are infinite cardinals. If $\mathbb{B}$ is γ-stable and δ-stable then $\gamma = \delta$.*

◁ If $C_\#(Q, l^1(\Gamma))$ and $C_\#(Q, l^1(\Delta))$ are lattice $\mathbb{B}$-isometric then $\mathbb{V}^{(\mathbb{B})} \models$"$l^1(\gamma^\wedge)$ and $l^1(\delta^\wedge)$ are lattice isometric" and thus $\mathbb{V}^{(\mathbb{B})} \models \operatorname{card}(\gamma^\wedge) = \operatorname{card}(\delta^\wedge)$. It remains to observe that $\mathbb{B}$ is γ-stable (δ-stable) if and only if $\mathbb{V}^{(\mathbb{B})} \models \operatorname{card}(\gamma^\wedge) = \gamma^\wedge$ (respectively $\operatorname{card}(\delta^\wedge) = \delta^\wedge$). ▷

Theorem 5.4. *Let X be a centrally atomic injective Banach lattice. Then there is a set of cardinals Γ and a stable partition of unity $(\pi_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{B} = \mathbb{M}(X)$ such that the following lattice $\mathbb{B}$-isometry holds:*

$$X \simeq_{\mathbb{B}} \Big(\sum\nolimits_{\gamma\in\Gamma}^{\oplus} l^1\big(\gamma, \Lambda_\gamma\big)\Big)_{l^\infty},$$

where $\Lambda_\gamma = \pi_\gamma\Lambda$ $(\gamma \in \Gamma)$. If a partition of unity $(\rho_\delta)_{\delta\in\Delta}$ in $\mathbb{B}$ satisfies the same conditions as $(\pi_\gamma)_{\gamma\in\Gamma}$, then $\Gamma = \Delta$, and $\pi_\gamma = \rho_\gamma$ for all $\gamma \in \Gamma$.

◁ The required representation follows from Proposition 4.2 and Lemma 5.2.

Assume now that a partition of unity $(\rho_\delta)_{\delta\in\Delta}$ in $\mathbb{B}$ satisfies the same conditions as $(\pi_\gamma)_{\gamma\in\Gamma}$. Fix $\delta \in \Delta$ and put $\sigma_{\gamma\delta} := \pi_\gamma\rho_\delta$ for arbitrary $\gamma \in \Gamma$. If $\sigma_{\gamma\delta} \neq 0$, then the injective Banach lattices $l^1(\gamma, \sigma_{\gamma\delta}\Lambda)$ and $l^1(\delta, \sigma_{\gamma\delta}\Lambda)$ are lattice $[\mathbb{O}, \sigma_{\delta\gamma}]$-isometric to the same band $\sigma_{\delta\gamma}X$. By Lemma 5.3 $\gamma = \delta$ and thus $\Delta \subset \Gamma$ and $\rho_\delta \leqslant \pi_\gamma$ for all $\delta \in \Delta$. Similarly, $\Gamma \subset \Delta$ and $\rho_\delta \geqslant \pi_\gamma$ for all $\gamma \in \Gamma$. ▷

Remark 5.5. Let Q be the Stone representation space of $\mathbb{B}$. Corollary 4.4 enables us to replace $l^1\big(\gamma, \Lambda_\gamma\big)$ by $C_\#(Q_\gamma, l^1(\gamma))$ in Theorem 5.4 with a stable partition of unity $(Q_\gamma)_{\gamma\in\Gamma}$ in he Boolean algebra of clopen subsets of Q. Moreover, if some partition of unity $(P_\delta)_{\delta\in\Delta}$ satisfies the same conditions, then $\Gamma = \Delta$, and $P_\gamma = Q_\gamma$ for all $\gamma \in \Gamma$.

Corollary 5.6. *Let X be an injective Banach lattice and Q the Stone representation space of $\mathbb{B} = \mathbb{M}(X)$. If X is $\mathbb{B}$-separable, then X is lattice $\mathbb{B}$-isometric to $C_\#(Q, l^1)$, $l^1 = l^1(\omega)$.*

◁ In Theorem 5.4 each component $l^1\big(\gamma, \Lambda_\gamma\big)$ is $\mathbb{B}_\gamma$-separable and hence its Boolean valued representation is a separable Banach lattice which is lattice isometric to the internal Banach lattice $l^1(\omega^\wedge)$. It follows that $l^1\big(\gamma, \Lambda_\gamma\big)$ is lattice $\mathbb{B}_\gamma$-isometric to $C_\#(Q_\gamma, l^1)$ for all $\gamma \in \Gamma$ by Proposition 4.1 and Corollary 4.4. From this it is obvious that X is $\mathbb{B}$-isometric to $C_\#(Q, l^1)$. ▷

Proposition 5.7. *A $\mathbb{B}$-cyclic Banach lattice is atomic if and only if it is $\mathbb{B}$-atomic and the Boolean algebra $\mathbb{B}$ is atomic.*

◁ The complete Boolean algebra $\mathbb{B}$ is atomic if and only if $\mathbb{B} = \mathscr{P}(A)$ for some set A and then X is the l^∞-sum of a family of Banach lattices $(X_a)_{a\in A}$. This l^∞-sum is evidently atomic if and only if X_a is atomic for all $a \in A$. ▷

The following corollary should be compared with [7, Theorem 5.6].

Corollary 5.8. *An injective Banach lattice X is atomic if and only if there is a set of cardinals Γ such that the following lattice isometry holds:*

$$X \simeq \Big(\sum\nolimits_{\gamma\in\Gamma}^{\oplus} l^1(\gamma)\Big)_{l^\infty}.$$

◁ In Remark 5.5 each Q_γ is a one-point space by Proposition 5.8 and hence $C_\#\big(Q_\gamma, l^1(\gamma)\big)$ is lattice isometric to $l^1(\gamma)$. ▷

Definition 5.9. The partition of unity $(\pi_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{B} = \mathbb{M}(X)$ satisfying the claim of Theorem 5.4 is called the *decomposition series* of X and is denoted by $d(X)$. Say that the decomposition series $d(X) = (\pi_\gamma)_{\gamma\in\Gamma}$ and $d(Y) = (\rho_\gamma)_{\gamma\in\Gamma}$ of centrally atomic injective Banach lattices X and Y are *congruent* if there exists a Boolean isomorphism τ from $\mathbb{M}(X)$ onto $\mathbb{M}(Y)$ such that $\tau(\pi_\gamma) = \rho_\gamma$ for all $\gamma \in \Gamma$.

Theorem 5.10. *Centrally atomic injective Banach lattices X and Y are lattice isometric if and only if the Boolean algebras $\mathbb{M}(X)$ and $\mathbb{M}(Y)$ are isomorphic and the decomposition series $d(X)$ and $d(Y)$ are congruent.*

$\lhd$ *Sufficiency.* Let X and Y be centrally atomic injective Banach lattices with $d(X) = (\pi_\gamma)_{\gamma\in\Gamma}$ and $d(Y) = (\rho_\gamma)_{\gamma\in\Gamma}$ and let $\mathscr{X}$ and $\mathscr{Y}$ be their respective Boolean valued representations. We identify X and Y with $\mathscr{X}{\Downarrow}$ and $\mathscr{Y}{\Downarrow}$, respectively. Denote $\mathbb{B} := \mathbb{M}(X)$ and $\mathbb{D} := \mathbb{M}(Y)$ and assume that there exists a Boolean isomorphism τ from $\mathbb{B}$ onto $\mathbb{D}$ such that $\tau(\pi_\gamma) = \rho_\gamma$ for all $\gamma \in \Gamma$. Recall that there is a bijective mapping $\tau^* : \mathbb{V}^{(\mathbb{B})} \to \mathbb{V}^{(\mathbb{D})}$ such that a ZFC-formula $\varphi(x_1, \dots, x_n)$ is true within $\mathbb{V}^{(\mathbb{B})}$ if and only if $\varphi(\tau^* x_1, \dots, \tau^* x_n)$ is true within $\mathbb{V}^{(\mathbb{D})}$ for all $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$, see [3, 1.3.1, 1.3.2, and 1.3.5 (2)]. It follows that $\tau^*(\mathscr{X})$ is an atomic injective Banach lattice within $\mathbb{V}^{(\mathbb{D})}$. Moreover, the mapping $x \mapsto \tau^*(x)$ $(x \in \mathscr{X}{\Downarrow})$ ia a lattice isometry from $\mathscr{X}{\Downarrow}$ onto $\tau^*(\mathscr{X}){\Downarrow}$. If $\alpha = \mathrm{card}(\mathrm{at}_1(\mathscr{X}))$ and $\beta = \mathrm{card}(\mathrm{at}_1(\mathscr{Y}))$, then $\tau^*(\alpha) = \mathrm{mix}_{\gamma\in\Gamma}\, \tau(\pi_\gamma)\gamma^\wedge$ and $\beta = \mathrm{mix}_{\gamma\in\Gamma}\, \rho\gamma^\wedge$, so that $\beta = \tau^*(\alpha)$. By [3, 1.3.5 (2)] we have $\tau^*(\alpha) = \mathrm{card}(\mathrm{at}_1(\tau^*(\mathscr{X})))$ and $\mathrm{card}(\mathrm{at}_1(\mathscr{Y})) = \mathrm{card}(\mathrm{at}_1(\tau^*(\mathscr{X})))$. It follows that $\tau^*(\mathscr{X})$ and $\mathscr{Y}$ are lattice isometric and hence $\tau^*(\mathscr{X}){\Downarrow}$ and $\mathscr{Y}{\Downarrow}$ are lattice $\mathbb{B}$-isometric.

Necessity. Suppose that h is a lattice isomorphism from X onto Y. Then the mapping τ from $\mathbb{B}$ onto $\mathbb{D}$ defined by $\tau(\pi) = h \circ \tau \circ h^{-1}$ is a Boolean isomorphism. Moreover, $h(\mathbb{B}\text{-}\mathrm{at}_1(\pi X)) = \mathbb{B}\text{-}\mathrm{at}_1(\tau(\pi)Y)$. Now it can be easily verified that πX is $([\mathbb{O}, \pi], \gamma)$-atomic if and only if $\tau(\pi)Y$ is $([\mathbb{O}, \tau(\pi)], \gamma)$-atomic. It follows that $d(X)$ and $d(Y)$ are congruent. $\rhd$

Corollary 5.11. *Let X be a centrally atomic injective Banach lattice. Then there is a family of Stonean spaces $(Q_\gamma)_{\gamma\in\Gamma}$, with Γ a set of cardinals, such that Q_γ is γ-stable for all $\gamma \in \Gamma$ and the following lattice $\mathbb{B}$-isometry holds:*

$$X \simeq_{\mathbb{B}} \Big(\sum\nolimits_{\gamma\in\Gamma}^{\oplus} C_{\#}\big(Q_\gamma, l^1(\gamma)\big) \Big)_{l^\infty}.$$

If some family $(P_\delta)_{\delta\in\Delta}$ of Stonean spaces satisfies the above conditions, then $\Gamma = \Delta$, and P_γ is homeomorphic with Q_γ for all $\gamma \in \Gamma$.

$\lhd$ This is immediate from Theorem 5.10 and since Corollary 4.4 (see Remark 5.5). $\rhd$

Definition 5.12. The *second* $\mathbb{B}$*-dual* of a $\mathbb{B}$-cyclic Banach space is defined by $X^{\#\#} := (X^{\#})^{\#} := \mathscr{L}_{\mathbb{B}}(X^{\#}, \Lambda)$. A $\mathbb{B}$-cyclic Banach space is said to be $\mathbb{B}$-*reflexive* if the image of X under the *canonical embedding* $X \to X^{\#\#}$ coincide with $X^{\#\#}$, see [3, p. 316].

Theorem 5.13. *Let X be a $\mathbb{B}$-reflexive injective Banach lattice with $\mathbb{B} = \mathbb{M}(X)$. Then there are a sequence of Stonean spaces $(Q_k)_{k\in\mathbb{N}}$, and an increasing sequence of naturals (n_k) such that the following lattice $\mathbb{B}$-isometry holds:*

$$X \simeq \Big(\sum\nolimits_{k\in\mathbb{N}}^{\oplus} C_{\#}\big(Q_k, l^1(n_k)\big) \Big)_{l^\infty}.$$

If some family $(P_k)_{k\in\mathbb{N}}$ of Stonean spaces satisfies the above conditions, then Q_k and P_k are homeomorphic for all $k \in \mathbb{N}$.

$\lhd$ Again identify X with $\mathscr{X}{\Downarrow}$, where $\mathscr{X}$ is an AL-space in $\mathbb{V}^{(\mathbb{B})}$. It follows from Theorem [3, Theorem 5.8.12] that $\mathscr{X}^*{\Downarrow} = \mathscr{X}{\Downarrow}^{\#}$ and $\mathscr{X}^{**}{\Downarrow} = \mathscr{X}{\Downarrow}^{\#\#}$. Therefore, X is $\mathbb{B}$-reflexive if and only if $[\![\mathscr{X} \text{ is reflexive}]\!] = \mathbb{1}$. Since a reflexive AL-space is finite-dimensional, we have

$$\mathbb{1} = [\![(\exists\, n \in \mathbb{N}^\wedge)\, \dim(\mathscr{X}) = n]\!] = \bigvee\nolimits_{n\in\mathbb{N}} [\![\dim(\mathscr{X}) = n^\wedge]\!].$$

This relation enables us to choose a countable partition of unity (b_n) in $\mathbb{B}$ such that $b_n \leqslant [\![\mathscr{X}$ is a $n^\wedge$-dimensional AL-space$]\!]$. Pick the sequence (n_k) of indices of nonzero projections in (b_n) and denote by Q_k the Stonean space of a Boolean algebra $\mathbb{B}_k := [\mathbb{O}, b_{n_k}]$. Now, by the Transfer Principle we conclude that $\mathbb{V}^{(\mathbb{B}_k)} \models$ "$b_{n_k} \wedge \mathscr{X}$ is lattice isometric to $l^1(n_k^\wedge)$". The proof is concluded with the help of Theorem 5.10 taking into consideration that for each finite cardinal γ every complete Boolean algebra is γ-*stable* and $\gamma^\wedge$ is a finite cardinal within $\mathbb{V}^{(\mathbb{B})}$. $\triangleright$

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АТОМИЧНОСТЬ В ИНЪЕКТИВНЫХ БАНАХОВЫХ РЕШЕТКАХ

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Цель заметки — рассмотреть булевозначную интерпретацию понятия атомической банаховой решетки и дать полное описание соответствующего класса инъективных банаховых решеток.

Ключевые слова: инъективная банахова решетка, атомическая банахова решетка, булевозначное представление, классификация.