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EXPONENTIAL STABILITY FOR A SWELLING POROUS-HEAT SYSTEM WITH THERMODIFFUSION EFFECTS AND DELAY

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Abstract. In the present work, we consider a one-dimensional swelling porous-heat system with single time-delay in a bounded domain under Dirichlet-Neumann boundary conditions subject to thermodiffusion effects and frictional damping to control the delay term. The coupling gives new contributions to the theory associated with asymptotic behaviors of swelling porous-heat. At first, we state and prove the well-posedness of the solution of the system by the semigroup approach using Lumer-Philips theorem under suitable assumption on the weight of the delay. Then, we show that the considered dissipation in which we depended on are strong enough to guarantee an exponential decay result by using the energy method that consists to construct an appropriate Lyapunov functional based on the multiplier technique, this result is obtained without the equal-speed requirement. Our result is new and an extension of many other works in this area.

Keywords: swelling porous, well-posedness, thermodiffusion effects, delay term, exponential stability.

AMS Subject Classification: 93D20, 35B40, 35L90, 45K05.

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1. Introduction

In this paper, we study well-posedness and exponential stability for a swelling porous-heat system with thermodiffusion effects and delay. The system is written as

$$\begin{cases} \rho_1 u_{tt} - a_1 u_{xx} - a_2 \varphi_{xx} = 0, \\ \rho_2 \varphi_{tt} - a_3 \varphi_{xx} - a_2 u_{xx} - \gamma_1 \theta_x - \gamma_2 P_x + \mu_1 \varphi_t + \mu_2 \varphi_t(x, t-\tau) = 0, \\ c\theta_t + dP_t - k\theta_{xx} - \gamma_1 \varphi_{xt} = 0, \\ d\theta_t + rP_t - hP_{xx} - \gamma_2 \varphi_{xt} = 0, \end{cases} \tag{1}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, and we impose the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in (0,1), \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), & x \in (0,1), \\ \theta(x,0) = \theta_0(x),\ P(x,0) = P_0(x), & x \in (0,1), \\ \varphi_t(x,-t) = f_0(x,t), & t \in (0,\tau), \\ u(0,t) = \varphi(0,t) = \theta(0,t) = P(0,t) = 0 & (\forall\, t \geqslant 0), \\ u_x(1,t) = \varphi_x(1,t) = \theta(1,t) = P(1,t) = 0 & (\forall\, t \geqslant 0), \end{cases} \tag{2}$$

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where $u = u(x,t)$ is the displacement of the fluid and $\varphi = \varphi(x,t)$ is the elastic solid material; ρ_1 and ρ_2 are the densities of u and φ, respectively; $\theta = \theta(x,t)$ is the temperature difference and $P = P(x,t)$ is the chemical potential; k and h are heat and mass diffusion conductivity coefficients, respectively. The coefficients a_1, a_3 are positive constants and $a_2 \neq 0$ is a real number such that $a_1a_3 > a_2^2$. The coefficients μ_1 is positive constant and μ_2 is a real number. Here, we prove the well-posedness and stability results for the problem (1)–(2), under the assumption

$$\mu_1 > |\mu_2|. \tag{3}$$

The physical positive constants γ_1, γ_2, r, c and d satisfying

$$\lambda = rc - d^2 > 0. \tag{4}$$

Equations $(1)_{1,2}$ are results of the basic field equations for the theory of swelling of one-dimensional porous elastic soils, given by (see [1])

$$\begin{cases} \rho_1 u_{tt} = T_{1x} - P_1 + F_1, \\ \rho_2 \varphi_{tt} = T_{2x} - P_2 + F_2, \end{cases} \tag{5}$$

where T_i are the partial tensions, F_i are the external forces, and P_i are internal body forces associated with the dependent variables u and φ, respectively. We assume that the constitutive equations of partial tensions as follows

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}}_{M} \begin{pmatrix} u_x \\ \varphi_x \end{pmatrix}, \tag{6}$$

where M is a positive definite symmetric array, i. e., $a_2^2 < a_1a_3$, and the internal forces of the body are considered null, that is, $P_1 = P_2 = 0$. We finally chose

$$F_1 = 0 \quad \text{and} \quad F_2 = \gamma_1\theta_x + \gamma_2 P_x - \mu_1\varphi_t - \mu_2\varphi_t(x, t-\tau).$$

Time delay equations have a wide range of applications in the biological, mechanical social sciences, and many other modelling of the phenomena. It depend not only on the present state but also on some past occurrences. We know the dynamic systems with delay terms have become a major research subject in differential equation since the 1970s of the last century (e. g. [2–8]). It was shown that delay is a source of instability unless additional conditions or control terms are used (see [9]). On the other hand, it may not only destabilize a system which is asymptotically stable in the absence of delay, but it may also lead to will posedness (see [10, 11] and the references therein). Therefore, the stability issue of systems with delay plays great importance theoretical and practical in most of researches. In [8], the authors considered (5) by taking

$$P_1 = P_2 = 0, \quad F_1 = -\mu_1\varphi_t - \mu_2\varphi_t(x, t-\tau) \quad \text{and} \quad F_2 = 0,$$

they proved that the energy associated with the system is dissipative, and established the exponential stability of the system. Readers can consult [12–20] and the references therein for some other crucial results on the swelling porous system.

The purpose of this work is to study system (1)–(2), in introducing the delay term and thermodiffusion effects can make the problem different and crucial among the literature considered. The main features of this paper are summarized as follows. In Section 2, we adopt the semigroup method and Lumer–Philips theorem to obtain the well-posedness of system (1)–(2). In Section 3, we use the perturbed energy method and construct Lyapunov functional to prove the exponential stability of system (1)–(2).

2. Well-Posedness

In this section, we prove the existence and uniqueness of solutions for (1)–(2). As in [7], we introduce the new variable

$$z(x,\rho,t)=\varphi_t(x,t-\tau\rho),\qquad x\in(0,1),\ \rho\in(0,1),\ t>0. \tag{7}$$

Therefore, problem (1) takes the form

$$\begin{cases}\rho_1 u_{tt}-a_1u_{xx}-a_2\varphi_{xx}=0,\\ \rho_2\varphi_{tt}-a_3\varphi_{xx}-a_2u_{xx}-\gamma_1\theta_x-\gamma_2P_x+\mu_1\varphi_t+\mu_2z(x,1,t)=0,\\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0,\\ c\theta_t+dP_t-k\theta_{xx}-\gamma_1\varphi_{xt}=0,\\ d\theta_t+rP_t-hP_{xx}-\gamma_2\varphi_{xt}=0,\end{cases} \tag{8}$$

with the following initial and boundary conditions

$$\begin{cases}u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in(0,1),\\ \varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x), & x\in(0,1),\\ \theta(x,0)=\theta_0(x),\ P(x,0)=P_0(x), & x\in(0,1),\\ z(x,\rho,0)=f_0(x,\tau\rho), & (x,\rho)\in(0,1)\times(0,1),\\ z(x,0,t)=\varphi_t(x,t), & (x,t)\in(0,1)\times(0,+\infty),\\ u(0,t)=\varphi(0,t)=\theta(0,t)=P(0,t)=0 & (\forall\, t\geqslant 0),\\ u_x(1,t)=\varphi_x(1,t)=\theta(1,t)=P(1,t)=0 & (\forall\, t\geqslant 0).\end{cases} \tag{9}$$

Introducing the vector function $U=(u,u_t,z,\varphi,\varphi_t,\theta,P)^T$. Then system (8)–(9) can be written as

$$\begin{cases}U'(t)=\mathscr{A}U(t),\ t>0,\\ U(0)=U_0=(u_0,u_1,\varphi_0,\varphi_1,f_0,\theta_0,P_0)^T,\end{cases} \tag{10}$$

where the operator $\mathscr{A}$ is defined by

$$\mathscr{A}\begin{pmatrix}u\\u_t\\\varphi\\\varphi_t\\z\\\theta\\P\end{pmatrix}=\begin{pmatrix}u_t\\ \frac{1}{\rho_1}\left[a_1u_{xx}+a_2\varphi_{xx}\right]\\ \varphi_t\\ \frac{1}{\rho_2}\left[a_3\varphi_{xx}+a_2u_{xx}+\gamma_1\theta_x+\gamma_2P_x-\mu_1\varphi_t-\mu_2z(x,1,t)\right]\\ -\frac{1}{\tau}z_\rho\\ \left(\frac{rk}{\lambda}\right)\theta_{xx}-\left(\frac{hd}{\lambda}\right)P_{xx}+\left(\frac{r\gamma_1-d\gamma_2}{\lambda}\right)\varphi_{tx}\\ \left(\frac{ch}{\lambda}\right)P_{xx}-\left(\frac{kd}{\lambda}\right)\theta_{xx}+\left(\frac{c\gamma_2-d\gamma_1}{\lambda}\right)\varphi_{tx}\end{pmatrix}.$$

Now, the energy space is defined by

$$\mathscr{H}=H^1_*(0,1)\times L^2(0,1)\times H^1_*(0,1)\times L^2(0,1)\times L^2\left((0,1),L^2(0,1)\right)\times L^2(0,1)\times L^2(0,1),$$

where

$$H^1_*(0,1) = \{f \in H^1(0,1);\ f(0) = 0\}.$$

Let

$$U = (u, u_t, \varphi, \varphi_t, z, \theta, P)^T, \quad \overline{U} = \left(\bar{u}, \bar{u}_t, \bar{\varphi}, \bar{\varphi}_t, \bar{z}, \bar{\theta}, \bar{P}\right)^T.$$

Then, for a positive constant ξ satisfying

$$\tau|\mu_2| \leqslant \xi \leqslant \tau(2\mu_1 - |\mu_2|), \tag{11}$$

we define the inner product in $\mathscr{H}$ as follows

$$\left\langle U, \overline{U}\right\rangle_{\mathscr{H}} = \int_0^1 \left[\rho_1 u_t \bar{u}_t + a_1 u_x \bar{u}_x + \rho_2 \varphi_t \bar{\varphi}_t + a_3 \varphi_x \bar{\varphi}_x + a_2 \left(u_x \bar{\varphi}_x + \varphi_x \bar{u}_x\right)\right] dx$$
$$+ \int_0^1 \left[c\theta\bar{\theta} + d\left(P\bar{\theta} + \theta\bar{P}\right) + rP\bar{P}\right] dx + \xi \int_0^1 \int_0^1 z\,(x,\rho)\,\bar{z}\,(x,\rho)\,d\rho dx.$$

The domain of $\mathscr{A}$ is

$$D(\mathscr{A}) = \{U \in \mathscr{H} \mid u, \varphi \in H^2_*(0,1),\ u_t, \varphi_t \in H^1_*(0,1),$$
$$\theta, P \in H^1_0(0,1),\ z, z_\rho \in L^2\big((0,1), L^2(0,1)\big)\},$$

where

$$H^2_*(0,1) = \{f \in H^2(0,1);\ f(0) = f_x(1) = 0\}.$$

Clearly, $D(\mathscr{A})$ is dense in $\mathscr{H}$.

We have the following existence and uniqueness result.

Theorem 1. *Assume that $U_0 \in \mathscr{H}$ and (4) holds, then problem (7)–(8) exists a unique solution $U \in C(\mathbb{R}^+; \mathscr{H})$. Moreover, if $U_0 \in D(\mathscr{A})$, then*

$$U \in C\left(\mathbb{R}^+; D(\mathscr{A})\right) \cap C^1\left(\mathbb{R}^+; \mathscr{H}\right).$$

$\lhd$ To obtain the above result, we need to prove that $\mathscr{A} : D(\mathscr{A}) \to \mathscr{H}$ is a maximal monotone operator. For this purpose, we need the following two steps: $\mathscr{A}$ is dissipative and $Id - \mathscr{A}$ is surjective.

Step 1. $\mathscr{A}$ is dissipative.

For any $U = (u, u_t, \varphi, \varphi_t, z, \theta, P)^T \in D(\mathscr{A})$, by using the inner product and integrating by parts, we obtain

$$\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = -k\int_0^1 \theta_x^2\,dx - h\int_0^1 P_x^2\,dx - \left(\mu_1 - \frac{\xi}{2\tau}\right)\int_0^1 \varphi_t^2\,dx$$
$$- \mu_2 \int_0^1 z(x,1,t)\,\varphi_t\,dx - \frac{\xi}{2\tau}\int_0^1 z^2(x,1,t)\,dx.$$

Using Young's inequality, we obtain

$$-\mu_2 \int_0^1 z(x,1,t)\,\varphi_t\,dx \leqslant \frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,dx + \frac{|\mu_2|}{2}\int_0^1 \varphi_t^2\,dx.$$

Therefore, from the assumption (11), we have

$$\langle \mathscr{A}U, U\rangle_{\mathscr{H}} \leqslant -k\int_0^1 \theta_x^2\,dx - \left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^1 \varphi_t^2\,dx \\ - h\int_0^1 P_x^2\,dx - \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^1 z^2(x,1,t)\,dx \leqslant 0.$$

Consequently, $\mathscr{A}$ is a dissipative operator.

STEP 2. $Id - \mathscr{A}$ is surjective.

To prove that the operator $Id - \mathscr{A}$ is surjective, that is, for any $F = (f_1, \ldots, f_7)^T \in \mathscr{H}$, there exists $U = (u, u_t, \varphi, \varphi_t, z, \theta, P)^T \in D(\mathscr{A})$ satisfying

$$(Id - \mathscr{A})U = F, \tag{12}$$

which is equivalent to

$$\begin{cases} u - u_t = f_1, \\ \rho_1 u_t - a_1 u_{xx} - a_2\varphi_{xx} = \rho_1 f_2, \\ \varphi - \varphi_t = f_3, \\ \rho_2\varphi_t - a_3\varphi_{xx} - a_2 u_{xx} - \gamma_1\theta_x - \gamma_2 P_x + \mu_1\varphi_t + \mu_2 z(x,1,t) = \rho_2 f_4, \\ \tau z + z_\rho = \tau f_5, \\ \lambda\theta - rk\theta_{xx} + hdP_{xx} - (r\gamma_1 - d\gamma_2)\,\varphi_{tx} = \lambda f_6, \\ \lambda P - chP_{xx} + kd\theta_{xx} - (c\gamma_2 - d\gamma_1)\,\varphi_{tx} = \lambda f_7. \end{cases} \tag{13}$$

Suppose that we have found u and φ with the appropriate regularity. Therefore, the first and the third equations in (13) give

$$\begin{cases} u_t = u - f_1, \\ \varphi_t = \varphi - f_3. \end{cases} \tag{14}$$

It is clear that $u_t \in H_*^1(0,1)$ and $\varphi_t \in H_*^1(0,L)$.

We note that the fifth equation in (13) with $z(x,0,t) = \varphi_t(x,t)$, has a unique solution

$$z(x,\rho,t) = \varphi(x)e^{-\tau\rho} - f_3(x)e^{-\tau\rho} + \tau e^{-\tau\rho}\int_0^\rho e^{\tau s} f_5(x,s)\,ds, \tag{15}$$

clearly, $z, z_\rho \in L^2((0,1)\times(0,1))$.

By using (13), (14) and (15) the functions (u, φ, θ, P) satisfy the following system

$$\begin{cases} \rho_1 u - a_1 u_{xx} - a_2\varphi_{xx} = g_1, \\ \eta\varphi - a_3\varphi_{xx} - a_2 u_{xx} - \gamma_1\theta_x - \gamma_2 P_x = g_2, \\ \lambda\theta - rk\theta_{xx} + hdP_{xx} - (r\gamma_1 - d\gamma_2)\varphi_x = g_3, \\ \lambda P - chP_{xx} + kd\theta_{xx} - (c\gamma_2 - d\gamma_1)\varphi_x = g_4, \end{cases} \tag{16}$$

where

$$\begin{cases} \eta = \rho_2 + \mu_1 + \mu_2 e^{-\tau}, \\ g_1 = \rho_1 f_1 + \rho_1 f_2, \\ g_2 = \rho_2 f_4 + \eta f_3 - \mu_2 \tau e^{-\tau} \int\limits_0^1 e^{\tau s} f_5(x,s)\, ds, \\ g_3 = \lambda f_6 - (r\gamma_1 - d\gamma_2) f_{3x}, \\ g_4 = \lambda f_7 - (c\gamma_2 - d\gamma_1)\, f_{3x}. \end{cases}$$

We multiply $(16)_1$ by $\tilde{u}$, $(16)_2$ by $\tilde{\varphi}$, $(16)_3$ by $\frac{c}{\lambda}\tilde{\theta}$, $(16)_4$ by $\frac{r}{\lambda}\tilde{P}$, $(16)_3$ by $\frac{d}{\lambda}\tilde{P}$ and $(16)_4$ by $\frac{d}{\lambda}\tilde{\theta}$ and integrate their sum over $(0,1)$ to find the following variational formulation

$$\mathscr{B}\Big((u,\varphi,\theta,P)^T, \big(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{P}\big)^T\Big) = \mathscr{G}\big(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{P}\big)^T, \tag{17}$$

where $\mathscr{B} : [H^1_*(0,1)\times H^1_*(0,1)\times L^2(0,1)\times L^2(0,1)]^2 \longrightarrow \mathbb{R}$ is the bilinear form given by

$$\begin{aligned} \mathscr{B}\Big((u,\varphi,\theta,P)^T, \big(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{P}\big)^T\Big) &= \rho_1 \int\limits_0^1 u\tilde{u}\, dx + a_1 \int\limits_0^1 u_x \tilde{u}_x\, dx + a_2 \int\limits_0^1 \left(\varphi_x \tilde{u}_x + u_x \tilde{\varphi}_x\right) dx \\ + \eta \int\limits_0^1 \varphi\tilde{\varphi}\, dx + a_3 \int\limits_0^1 \varphi_x\tilde{\varphi}_x\, dx &+ c\int\limits_0^1 \theta\tilde{\theta}\, dx + k\int\limits_0^1 \theta_x\tilde{\theta}_x\, dx + r\int\limits_0^1 P\tilde{P}\, dx + h\int\limits_0^1 P_x\tilde{P}_x dx \\ + d\int\limits_0^1 \big(\theta\tilde{P} + P\tilde{\theta}\big)\, dx &+ \gamma_1 \int\limits_0^1 \big(\theta\tilde{\varphi}_x - \varphi_x\tilde{\theta}\big)\, dx + \gamma_2 \int\limits_0^1 \big(P\tilde{\varphi}_x - \varphi_x\tilde{P}\big)\, dx, \end{aligned}$$

and $\mathscr{G} : \left[H^1_*(0,1)\times H^1_*(0,1)\times L^2(0,1)\times L^2(0,1)\right] \longrightarrow \mathbb{R}$ is the linear form defined by

$$\mathscr{G}\big(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{P}\big)^T = \int\limits_0^1 g_1\tilde{u}\, dx + \int\limits_0^1 g_2\tilde{\varphi}\, dx + \frac{c}{\lambda}\int\limits_0^1 g_3\tilde{\theta}\, dx + \frac{r}{\lambda}\int\limits_0^1 g_4\tilde{P}\, dx + \frac{d}{\lambda}\int\limits_0^1 g_3\tilde{P}\, dx + \frac{d}{\lambda}\int\limits_0^1 g_4\tilde{\theta}\, dx.$$

It is easy to verify that $\mathscr{B}$ is continuous and coercive, and $\mathscr{G}$ is continuous. So applying the Lax–Milgram theorem, we deduce that for all $(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{P}) \in H^1_*(0,1)\times H^1_*(0,1)\times L^2(0,1)\times L^2(0,1)$, problem (17) admits a unique solution $(u,\varphi,\theta,P) \in H^1_*(0,1)\times H^1_*(0,1)\times L^2(0,1)\times L^2(0,1)$. The application of the classical regularity theory, it follows from (16) that $(u,\varphi,\theta,P) \in H^2_*(0,1)\times H^2_*(0,1)\times H^1_0(0,1)\times H^1_0(0,1)$. Hence, the operator $Id - \mathscr{A}$ is surjective. Consequently, the result of Theorem 1 follows from Lumer–Phillips theorem (see [21, 22]). ▷

3. Exponential Stability

In this section, we prove the exponential decay for problem (8)–(9). It will be achieved by using the energy method to produce a suitable Lyapunov functional. We define the energy functional $E(t)$ as

$$\begin{aligned} E(t) = \frac{1}{2}\int\limits_0^1 \left[\rho_1 u_t^2 + a_1 u_x^2 + \rho_2\varphi_t^2 + a_3\varphi_x^2 + 2a_2 u_x\varphi_x + c\theta^2 + 2d\theta P + rP^2\,\right] dx \\ + \frac{\xi}{2}\int\limits_0^1\int\limits_0^1 z^2(x,\rho,t)\, d\rho dx. \end{aligned} \tag{18}$$

Noting (4), we have for $\theta, P \neq 0$,

$$c\theta^2 + 2d\theta P + rP^2 = \frac{\lambda}{r}\,\theta^2 + \left(\frac{d}{\sqrt{r}}\,\theta + \sqrt{r}P\right)^2 > 0,$$

then we get that the energy $E(t)$ is positive.

The stability result reads as follows.

Theorem 2. *Let $(u, z, \varphi, \theta, P)$ be the solution of* (8)–(9) *and* (4) *holds. Then there exist two positive constants k_0 and k_1, such that*

$$E(t) \leqslant k_0 e^{-k_1 t} \quad (\forall\, t \geqslant 0). \tag{19}$$

Before defining a Lyapunov functional, we need some lemmas as follows.

Lemma 1. *Let $(u, z, \varphi, \theta, P)$ be the solution of* (7)–(8) *and* (4) *holds. Then, the energy functional, defined by equation* (18), *satisfies*

$$\frac{d}{dt}\,E(t) \leqslant -k\int_0^1 \theta_x^2\,dx - h\int_0^1 P_x^2\,dx - C_1\int_0^1 \varphi_t^2\,dx - C_2\int_0^1 z^2(x,1,t)\,dx \leqslant 0, \tag{20}$$

where

$$C_1 = \mu_1 - \frac{|\mu_2|}{2} - \frac{\xi}{2\tau} \geqslant 0, \quad C_2 = \frac{\xi}{2\tau} - \frac{|\mu_2|}{2} \geqslant 0.$$

$\lhd$ Multiplying $(8)_1$, $(8)_3$, $(8)_4$ and $(8)_5$ by u_t, φ_t, θ and P, respectively, and integrating over $(0,1)$ with respect to x, using integration by parts and the boundary conditions, we obtain

$$\begin{aligned}\frac{d}{dt}&\left[\frac{1}{2}\int_0^1 \left(\rho_1 u_t^2 + a_1 u_x^2 + \rho_2\varphi_t^2 + a_3\varphi_x^2 + 2a_2 u_x\varphi_x + c\theta^2 + 2d\theta P + rP^2\right)dx\right] \\ &= -k\int_0^1 \theta_x^2\,dx - h\int_0^1 P_x^2\,dx - \mu_1\int_0^1 \varphi_t^2\,dx - \mu_2\int_0^1 \varphi_t z(x,1,t)\,dx.\end{aligned} \tag{21}$$

On the other hand, multiplying $(8)_2$ by $\frac{\xi}{\tau}z(x,\rho,t)$ and integrating over $(0,1)\times(0,1)$, and recalling $z(x,0,t) = \varphi_t$, we obtain

$$\frac{\xi}{2}\,\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx = \frac{\xi}{2\tau}\int_0^1 \varphi_t^2\,dx - \frac{\xi}{2\tau}\int_0^1 z^2(x,1,t)\,dx. \tag{22}$$

A combination of (21) and (22) gives

$$\begin{aligned}\frac{d}{dt}\,E(t) = -k\int_0^1 \theta_x^2\,dx - h\int_0^1 P_x^2\,dx - \left(\mu_1 - \frac{\xi}{2\tau}\right)\int_0^1 \varphi_t^2\,dx \\ - \frac{\xi}{2\tau}\int_0^1 z^2(x,1,t)\,dx - \mu_2\int_0^1 \varphi_t z(x,1,t)\,dx.\end{aligned} \tag{23}$$

Now, estimate the last term of the right-hand side of (23) as follows

$$-\mu_2 \int_0^1 z(x,1,t)\varphi_t\,dx \leqslant \frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,dx + \frac{|\mu_2|}{2}\int_0^1 \varphi_t^2\,dx. \tag{24}$$

Substituting (24) into (23), and using (11), we obtain (20), which completes the proof. ▷

Lemma 2. *Let (u,z,φ,θ,P) be the solution of* (8)–(9)*. Then the functional*

$$L_1(t) = -\rho_1 \int_0^1 u u_t\,dx,$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$L_1'(t) \leqslant -\rho_1 \int_0^1 u_t^2\,dx + \left(a_1 + \frac{a_2^2}{4\varepsilon_1}\right)\int_0^1 u_x^2\,dx + \varepsilon_1 \int_0^1 \varphi_x^2\,dx. \tag{25}$$

◁ By differentiating $L_1(t)$ with respect to t, using $(8)_1$ and integrating by parts, we obtain

$$L_1'(t) = -\rho_1 \int_0^1 u_t^2\,dx + a_1 \int_0^1 u_x^2\,dx + a_2 \int_0^1 u_x\varphi_x\,dx,$$

then, by Young's inequality, we obtain the result. ▷

Lemma 3. *Let (u,z,φ,θ,P) be the solution of* (8)–(9)*. Then the functional*

$$L_2(t) = a_1\rho_2 \int_0^1 \varphi\varphi_t\,dx - a_2\rho_1 \int_0^1 \varphi u_t\,dx,$$

satisfies, for any $\varepsilon_2 > 0$, the estimate

$$\begin{aligned} L_2'(t) \leqslant -\frac{a}{2}\int_0^1 \varphi_x^2\,dx + C_1(\varepsilon_2)\int_0^1 \varphi_t^2\,dx + \varepsilon_2\int_0^1 u_t^2\,dx + \frac{2a_1^2\gamma_1^2}{a}\int_0^1 \theta_x^2\,dx \\ + \frac{2a_1^2\gamma_2^2}{a}\int_0^1 P_x^2\,dx + \frac{2a_1^2\mu_2^2}{a}\int_0^1 z^2(x,1,t)\,dx, \end{aligned} \tag{26}$$

where

$$a = a_1a_3 - a_2^2 > 0, \quad C_3(\varepsilon_2) = a_1\rho_2 + \frac{2a_2^2\mu_1^2}{a} + \frac{a_2^2\rho_1^2}{4\varepsilon_2}.$$

◁ By differentiating $L_2(t)$ with respect to t, using the equations $(8)_1$ and $(8)_2$, and integrating by parts, we obtain

$$\begin{aligned} L_2'(t) = -a\int_0^1 \varphi_x^2\,dx + a_1\rho_2\int_0^1 \varphi_t^2\,dx - a_2\rho_1\int_0^1 \varphi_t u_t\,dx + \gamma_1 a_1\int_0^1 \varphi\theta_x\,dx \\ + \gamma_2 a_1\int_0^1 \varphi P_x\,dx - \mu_1 a_1\int_0^1 \varphi\varphi_t\,dx - \mu_2 a_1\int_0^1 \varphi z(x,1,t)\,dx, \end{aligned}$$

where $a = a_3a_1 - a_2^2 > 0$. Using Young's and Poincaré inequalities, estimate (26) is established. $\rhd$

Lemma 4. *Let $(u, z, \varphi, \theta, P)$ be the solution of (8)–(9) and (3) holds. Then the functional*

$$L_3(t) = \frac{a_1\rho_2}{a_2}\int_0^1 \varphi_t u\,dx - \frac{a_3\rho_1}{a_2}\int_0^1 u_t\varphi\,dx,$$

satisfies, for any $\varepsilon_3 > 0$, the estimate

$$\begin{aligned} L_3'(t) \leqslant {} & -\frac{a_1}{2}\int_0^1 u_x^2\,dx + a_3\int_0^1 \varphi_x^2\,dx + \frac{2a_1\gamma_1^2}{a_2^2}\int_0^1 \theta_x^2\,dx + \frac{2a_1\gamma_2^2}{a_2^2}\int_0^1 P_x^2\,dx \\ & + C_2(\varepsilon_3)\int_0^1 \varphi_t^2\,dx + \frac{2a_1\mu_2^2}{a_2^2}\int_0^1 z^2(x,1,t)\,dx + \varepsilon_3\int_0^1 u_t^2\,dx, \end{aligned} \tag{27}$$

where

$$C_4(\varepsilon_3) = \frac{2a_1\mu_1^2}{a_2^2} + \frac{1}{4\varepsilon_3}\left(\frac{a_1\rho_2 - a_3\rho_1}{a_2}\right)^2.$$

$\lhd$ By differentiating $L_3(t)$ with respect to t, using the equations $(8)_1$ and $(8)_2$, and integrating by parts, we obtain

$$\begin{aligned} L_3'(t) = {} & -a_1\int_0^1 u_x^2\,dx + a_3\int_0^1 \varphi_x^2\,dx + \frac{\gamma_1 a_1}{a_2}\int_0^1 \theta_x u\,dx + \frac{\gamma_2 a_1}{a_2}\int_0^1 P_x u\,dx \\ & - \frac{\mu_1 a_1}{a_2}\int_0^1 \varphi_t u\,dx - \frac{\mu_2 a_1}{a_2}\int_0^1 z(x,1,t)u\,dx + \left(\frac{a_1\rho_2 - a_3\rho_1}{a_2}\right)\int_0^1 \varphi_t u_t\,dx. \end{aligned}$$

Using Young's and Poincaré inequalities, estimate (27) is established. $\rhd$

Lemma 5. *Let $(u, z, \varphi, \theta, P)$ be the solution of (8)–(9) and (3) holds. Then the functional*

$$L_4(t) = \int_0^1\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\,d\rho dx,$$

satisfies, for some positive constants n_1 and n_2, the estimate

$$L_4'(t) \leqslant -n_1\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx - n_2\int_0^1 z^2(x,1,t)\,dx + \frac{1}{\tau}\int_0^1 \varphi_t^2\,dx. \tag{28}$$

◁ By differentiating $L_4(t)$ with respect to t, and using the equation $(7)_3$, we obtain

$$
\begin{aligned}
L_4'(t) &= 2\int_0^1\int_0^1 e^{-2\tau\rho} z(x,\rho,t) z_t(x,\rho,t)\,d\rho dx = -\frac{2}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho} z(x,\rho,t) z_\rho(x,\rho,t)\,d\rho dx \\
&= -\frac{1}{\tau}\int_0^1\int_0^1 \frac{d}{d\rho}\left(e^{-2\tau\rho} z^2(x,\rho,t)\right) d\rho dx - 2\int_0^1\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\,d\rho dx \\
&\leqslant -n_1\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx - n_2\int_0^1 z^2(x,1,t)\,dx + \frac{1}{\tau}\int_0^1 \varphi_t^2\,dx,
\end{aligned}
$$

which gives the estimate (28). ▷

Now, we turn to prove our main result in this section.

◁ Proof of Theorem 2. We define the Lyapunov functional $\mathscr{L}(t)$ by

$$
\mathscr{L}(t) := NE(t) + \sum_{i=1}^{3} N_i L_i(t) + L_4(t),
$$

where N and N_i $(i = 1, 2, 3)$ are positive constants that will be chosen later.

By differentiating $\mathscr{L}(t)$, exploiting (20) and (25)–(28), we get

$$
\begin{aligned}
\mathscr{L}'(t) &\leqslant -[\rho_1 N_1 - \varepsilon_2 N_2 - \varepsilon_3 N_3]\int_0^1 u_t^2\,dx - \left[\frac{a_1}{2}N_3 - \left(a_1 + \frac{a_2^2}{4\varepsilon_1}\right)N_1\right]\int_0^1 u_x^2\,dx \\
&- \left[C_1 N - C_3(\varepsilon_2)N_2 - C_4(\varepsilon_3)N_3 - \frac{1}{\tau}\right]\int_0^1 \varphi_t^2\,dx - \left[\frac{a}{2}N_2 - a_3 N_3 - \varepsilon_1 N_1\right]\int_0^1 \varphi_x^2\,dx \\
&- \left[kN - \frac{2a_1^2\gamma_1^2}{a}N_2 - \frac{2a_1\gamma_1^2}{a_2^2}N_3\right]\int_0^1 \theta_x^2\,dx - \left[hN - \frac{2a_1^2\gamma_2^2}{a}N_2 - \frac{2a_1\gamma_2^2}{a_2^2}N_3\right]\int_0^1 P_x^2\,dx \\
&- n_1\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx - \left[C_2 N + n_2 - \frac{2a_1^2\mu_2^2}{a}N_2 - \frac{2a_1\mu_2^2}{a_2^2}N_3\right]\int_0^1 z^2(x,1,t)\,dx.
\end{aligned}
$$

At this point, taking $\varepsilon_i = 1/N_i$, $i = 1, 2, 3$. We then choose N_1 large enough so that $N_1 > 2/\rho_1$. After that, we select N_3 so that

$$
\frac{a_1}{2}N_3 - \left(a_1 + \frac{a_2^2 N_1}{4}\right)N_1 > 0.
$$

Then, we choose N_2 large enough so that

$$
\frac{a}{2}N_2 - a_3 N_3 - 1 > 0.
$$

Finally, we select N large enough so that

$$\begin{aligned} C_1N - C_3(\varepsilon_2)N_2 - C_4(\varepsilon_3)N_3 - \frac{1}{\tau} > 0, \\ kN - \frac{2a_1^2\gamma_1^2}{a} N_2 - \frac{2a_1\gamma_1^2}{a_2^2} N_3 > 0, \\ hN - \frac{2a_1^2\gamma_2^2}{a} N_2 - \frac{2a_1\gamma_2^2}{a_2^2} N_3 > 0, \\ C_2N + n_2 - \frac{2a_1^2\mu_2^2}{a} N_2 - \frac{2a_1\mu_2^2}{a_2^2} N_3 > 0. \end{aligned}$$

Consequently, from the above, we deduce that there exist a positive constant α_0 such that

$$\mathscr{L}'(t) \leqslant -\alpha_0 E(t). \tag{29}$$

On the hand, it is not hard to see that $\mathscr{L}(t) \sim E(t)$, i. e., there exist two positive constants α_1 and α_2 such that

$$\alpha_1 E(t) \leqslant \mathscr{L}(t) \leqslant \alpha_2 E(t) \quad (\forall\, t \geqslant 0). \tag{30}$$

Combining (29) and (30), we obtain that

$$\mathscr{L}'(t) \leqslant -k_1 \mathscr{L}(t) \quad (\forall\, t \geqslant 0), \tag{31}$$

where $k_1 = \frac{\alpha_0}{\alpha_2}$. A simple integration of (31) over $(0, t)$ yields

$$\mathscr{L}(t) \leqslant \mathscr{L}(0)e^{-k_1 t} \quad (\forall\, t \geqslant 0).$$

It gives the desired result of Theorem 2 when combined with the equivalence of $\mathscr{L}(t)$ and $E(t)$. $\rhd$

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ЭКСПОНЕНЦИАЛЬНАЯ УСТОЙЧИВОСТЬ ДЛЯ НАБУХАЮЩЕЙ ПОРИСТОЙ ТЕПЛОСИСТЕМЫ С ТЕРМОДИФФУЗИОННЫМИ ЭФФЕКТАМИ И ЗАПАЗДЫВАНИЕМ

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Аннотация. В настоящей работе рассматривается одномерная набухающая пористо-тепловая система в ограниченной области при граничных условиях Дирихле — Неймана с термодиффузионными эффектами и запаздыванием. Известно, что запаздывание без дополнительных предположений служит источником неустойчивости. Более того, введении запаздывания в асимтотически устойчивую систему может привести не только к потере устойчивости, но и к некорректно поставленной задаче. В этой связи исследование систем с запаздыванием на устойчивость имеет большое теоретическое и прикладное значение. Связанность системы вносит новый вклад в теорию, связанную с асимптотическим поведением набухания пористого тепла. Сначала мы формулируем и доказываем корректность решения системы полугрупповым подходом с использованием теоремы Люмера — Филипса при подходящем предположении о весе запаздывания. Затем получаем результат экспоненциального затухания, используя энергетический метод, основанный на методе умножения, в котором мы строим соответствующий функционал Ляпунова, этот результат получается без требования равной скорости. Наш результат является новым и является продолжением многих других работ в этой области.

Ключевые слова: набухание, пористость, термодиффузионные эффекты, запаздывающий член, экспоненциальная устойчивость.

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