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ON A NEW COMBINATION OF ORTHOGONAL POLYNOMIALS SEQUENCES

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Abstract. In this paper, we are interested in the following inverse problem. We assume that $\{P_n\}_{n\geqslant 0}$ is a monic orthogonal polynomials sequence with respect to a quasi-definite linear functional u and we analyze the existence of a sequence of orthogonal polynomials $\{Q_n\}_{n\geqslant 0}$ such that we have a following decomposition $Q_n(x) + r_nQ_{n-1}(x) = P_n(x) + s_nP_{n-1}(x) + t_nP_{n-2}(x) + v_nP_{n-3}(x)$, $n \geqslant 0$, when $v_nr_n \neq 0$, for every $n \geqslant 4$. Moreover, we show that the orthogonality of the sequence $\{Q_n\}_{n\geqslant 0}$ can be also characterized by the existence of sequences depending on the parameters r_n, s_n, t_n, v_n and the recurrence coefficients which remain constants. Furthermore, we show that the relation between the corresponding linear functionals is $k(x - c)u = (x^3 + ax^2 + bx + d)v$, where $c, a, b, d \in \mathbb{C}$ and $k \in \mathbb{C} \setminus \{0\}$. We also study some subcases in which the parameters r_n, s_n, t_n and v_n can be computed more easily. We end by giving an illustration for a special example of the above type relation.

Key words: orthogonal polynomials, linear functionals, inverse problem, Chebyshev polynomials.

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1. Introduction

Let $\mathscr{P}$ be the linear space of polynomials in one variable with complex coefficients and let $\mathscr{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the action of u in $\mathscr{P}'$ on f in $\mathscr{P}$ and by $(u)_n := \langle u, x^n\rangle$, $n \geqslant 0$, the moments of u with respect to the monomial sequence $\{x^n\}_{n\geqslant 0}$. When $(u)_0 = 1$, the linear functional u is said to be normalized. For our work we need to recall some operations in $\mathscr{P}'$ (see [1, 2]). For any u in $\mathscr{P}'$, any q in $\mathscr{P}$ and any complex numbers a, b, c with $a \neq 0$, let $Du = u'$, qu, h_au, τ_bu and σu be respectively the derivative, the left multiplication, the translation, the homothetic and the pair part of the linear functionals defined by duality:

$$\langle u', f\rangle := -\langle u, f'\rangle, \quad \langle qu, f\rangle := \langle u, qf\rangle, \quad \langle \tau_bu, f\rangle := \langle u, \tau_{-b}f\rangle = \langle u, f(x+b)\rangle,$$
$$\langle h_au, f\rangle := \langle u, h_af\rangle = \langle u, f(ax)\rangle, \quad \langle \sigma u, f\rangle := \langle u, \sigma f\rangle = \big\langle u, f\big(x^2\big)\big\rangle, \quad f \in \mathscr{P}.$$

The linear functional u is called regular (quasi-definite) if the leading principal submatrices $\mathscr{H}_n$ of the Hankel matrix $\mathscr{H} = (u_{i+j})_{i,j\geqslant 0}$ related to the moments $(u)_n = \langle u, x^n\rangle$, $n \geqslant 0$, are nonsingular, for each $n \geqslant 0$ [1].

Definition 1.1 [1]. A sequence of monic polynomials $\{P_n\}_{n\geqslant 0}$ is called *orthogonal* with respect to the linear functional u if the following orthogonality conditions hold

$$\langle u, P_n(x)P_m(x)\rangle = 0, \quad n \neq m,$$
$$\left\langle u, P_n^2(x)\right\rangle \neq 0, \quad n \geqslant 0,$$

where $\deg P_n = n$, for every $n \geqslant 0$.

In this way, $\{P_n\}_{n\geqslant 0}$ satisfies the following two order recurrence relation:

$$\begin{cases} P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \ n \geqslant 1, \\ P_0(x) = 1, \ P_1(x) = x - \beta_0, \end{cases}$$

where $\gamma_n \neq 0$, for each $n \geqslant 1$.

Let u and v be two regular linear functionals and let $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ be the corresponding sequences of monic orthogonal polynomials. Assume that there exist non-negative integer numbers M and N, and sequences of complex numbers $\{r_{i,n}\}_{n\geqslant 0}$ and $\{s_{k,n}\}_{n\geqslant 0}$ such that the structure relation

$$Q_n(x) + \sum_{i=1}^{M} r_{i,n}\, Q_{n-i}(x) = P_n(x) + \sum_{i=1}^{N} s_{i,n}\, P_{n-i}(x)$$

holds for $n \geqslant 0$. Further, assume that $r_{M,M+N} \neq 0$ and $s_{N,M+N} \neq 0$, $\det\left[\alpha_{ij}\right]_{i,j=1}^{M+N} \neq 0$, where the entries α_{ij} of the matrix are defined on the basis of $\{r_{i,n}\}_{n\geqslant 0}$ and $\{s_{k,n}\}_{n\geqslant 0}$. Then there exist two polynomials Φ and Ψ with $\deg \Phi = M$ and $\deg \Psi = N$ such that

$$\Phi(z)u = \Psi(z)v.$$

These polynomials Φ and Ψ can be constructed in an explicit way [3]. On the other hand, the converse result is also analyzed. A characterization theorem for the sequence $\{Q_n\}_{n\geqslant 0}$ to be orthogonal assuming $\{P_n\}_{n\geqslant 0}$ is orthogonal is obtained when $M = 0$ and $N = 1$, $M = 1$ and $N = 1$, $M = 0$ and $N = 2$, $M = 1$ and $N = 2$, $M = 0$ and $N = 3$, $M = 0$ and $N = k$ [4–8].

In this contribution, the main purpose is to analyze the inverse problem corresponding to the case $M = 1$ and $N = 3$, i. e.,

$$Q_n(x) + r_n Q_{n-1}(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x) + v_n P_{n-3}(x), \quad n \geqslant 0, \tag{1.1}$$

with the initial conditions $Q_0(x) = P_0(x) = 1$ and $Q_{-1}(x) = P_{-m}(x) = 0$, for $m \geqslant 1$, and where $\{r_n\}_{n\geqslant 0}$, $\{s_n\}_{n\geqslant 0}$, $\{t_n\}_{n\geqslant 0}$ and $\{v_n\}_{n\geqslant 0}$ are sequences of complex numbers with the initial conditions $r_0 = s_0 = t_0 = t_1 = v_0 = v_1 = v_2 = 0$ and $r_n v_n \neq 0$ when $n \geqslant 4$.

We provide necessary and sufficient conditions for the orthogonality of the monic polynomials sequence $\{Q_n\}_{n\geqslant 0}$ assuming the orthogonality of the sequence of monic polynomials $\{P_n\}_{n\geqslant 0}$. In addition, we establish a relation between the linear functionals u and v, respectively, corresponding to MOPS's $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ as $k(x-c)u = (x^3+ax^2+bx+d)v$ with $a, b, c, d \in \mathbb{C}$ and $k \in \mathbb{C} \setminus \{0\}$.

This paper is organized as follows. In Section 2, we develop some basic results and lemmas. Section 3, is devoted to find the characterizations of the orthogonality of the monic polynomials sequence $\{Q_n\}_{n\geqslant 0}$. Finally, we illustrate a special case of the above type relation.

2. 2–4 Type Relation

Let $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ be two sequences of monic orthogonal polynomials with respect to the regular functionals u and v respectively, where $\langle u,1\rangle = \langle v,1\rangle = 1$, let $\{\beta_n\}_{n\geqslant 0}$, $\{\gamma_n\}_{n\geqslant 1}$ and $\{\widetilde{\beta}_n\}_{n\geqslant 0}$, $\{\widetilde{\gamma}_n\}_{n\geqslant 1}$ be the corresponding sequences of recurrence coefficients characterizing $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ respectively. Suppose that these sequences are related by relation (1.1).

The initial conditions $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ and $v_4 r_4 \neq 0$ yield a relation between the linear functionals u and v such as

$$\phi u = \psi v,$$

where ϕ and ψ are polynomials of degree 1 and 3, respectively.

Firstly, if $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ and $r_4 \neq 0$, then there exists a complex number c such that

$$\big\langle (x-c)u, Q_4(x)\big\rangle = 0.$$

Moreover, $\langle (x-c)u, Q_n(x)\rangle = 0,\ \ n \geqslant 4$.

Indeed

$$\begin{aligned}
\langle (x-c)u, Q_0(x)\rangle &= \beta_0 - c,\\
\langle (x-c)u, Q_1(x)\rangle &= \gamma_1 + (s_1 - r_1)(\beta_0 - c),\\
\langle (x-c)u, Q_2(x)\rangle &= (s_2 - r_2)\gamma_1 + (t_2 - r_2(s_1 - r_1))(\beta_0 - c),\\
\langle (x-c)u, Q_3(x)\rangle &= (t_3 - r_3(s_2 - r_2))\gamma_1 + (v_3 - r_3(t_2 - r_2(s_1 - r_1)))(\beta_0 - c),\\
\langle (x-c)u, Q_4(x)\rangle &= (v_4 - r_4(t_3 - r_3(s_2 - r_2)))\gamma_1 - r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1)))(\beta_0 - c).
\end{aligned} \tag{2.1}$$

Then there exists c such that

$$\langle (x-c)u, Q_4(x)\rangle = 0.$$

This implies

$$c := \beta_0 - \frac{\gamma_1}{r_4}\,\frac{v_4 - r_4(t_3 - r_3(s_2 - r_2))}{v_3 - r_3(t_2 - r_2(s_1 - r_1))}. \tag{2.2}$$

Thus,

$$\langle (x-c)u, Q_n(x)\rangle = -r_n\langle (x-c)u, Q_{n-1}(x)\rangle, \quad n \geqslant 5.$$

On the other hand [2]

$$(x-c)u = \sum_{i=0}^{3} \frac{\langle (x-c)u, Q_i(x)\rangle}{\langle v, Q_i^2(x)\rangle}\, Q_i(x)v.$$

Therefore, if $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ and $v_4 r_4 \neq 0$, we see that the relation between u and v is

$$(x-c)u = q(x)v,$$

where q is a polynomial of exact degree 3.

Lemma 2.1. *Let $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ be two MOPSs with respect to the regular normalized linear functionals u and v respectively, where $\langle u,1\rangle = \langle v,1\rangle = 1$. Assume that there exist sequences of complex numbers $\{r_n\}_{n\geqslant 0}$, $\{s_n\}_{n\geqslant 0}$, $\{t_n\}_{n\geqslant 0}$ and $\{v_n\}_{n\geqslant 0}$ with the initial conditions $r_0 = s_0 = t_0 = t_1 = v_0 = v_1 = v_2 = 0$, such that the relation*

$$Q_n(x) + r_n Q_{n-1}(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x) + v_n P_{n-3}(x), \quad n \geqslant 0,$$

with the initial conditions $Q_0(x) = P_0(x) = 1$ and $Q_{-1}(x) = P_{-m}(x) = 0$, for $m = 1, 2, 3$, holds, for every $n \geqslant 0$. Then the following implications hold:

1. *If* $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ *and* $r_4 = 0$*, then* $v_n \neq r_n(t_{n-1} - r_{n-1}(s_{n-2} - r_{n-2}))$*, for* $n \geqslant 3$*, and* $r_n = 0$*, for every* $n \geqslant 4$*. In this case the relation* (1.1) *reduce to* 1–4 *type relation*

$$Q_n(x) = P_n(x) + a_n P_{n-1}(x) + b_n P_{n-2}(x) + c_n P_{n-3}(x), \quad n \geqslant 0$$

with

$$a_n := s_n - r_n, \quad n \geqslant 1,$$
$$b_n := t_n - r_n(s_{n-1} - r_{n-1}), \quad n \geqslant 2,$$
$$c_n := v_n - r_n(t_{n-1} - r_{n-1}(s_{n-2} - r_{n-2})), \quad n \geqslant 3.$$

2. *If* $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ *and* $r_4 \neq 0$*, then* $r_n \neq 0$*,* $n \geqslant 4$*.*
3. *If* $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ *and* $v_4 r_4 \neq 0$*, then* $v_n r_n \neq 0$*, for* $n \geqslant 4$*.*

Thus, in this case the relation (1.1) *is a non-degenerate* 2–4 *type relation.*

$\lhd$ We have

$$\begin{cases} \langle u, Q_1(x)\rangle = s_1 - r_1, \\ \langle u, Q_2(x)\rangle = t_2 - r_2(s_1 - r_1), \\ \langle u, Q_3(x)\rangle = v_3 - r_3(t_2 - r_2(s_1 - r_1)), \\ \langle u, Q_n(x)\rangle = -r_n\langle u, Q_{n-1}\rangle, \ n \geqslant 4. \end{cases} \tag{2.3}$$

If $v_3 = r_3(t_2 - r_2(s_1 - r_1))$, then we have the following cases:

i) $t_2 = r_2(s_1 - r_1)$ and $s_1 = r_1$.

ii) $t_2 = r_2(s_1 - r_1)$ and $s_1 \neq r_1$.

iii) $t_2 \neq r_2(s_1 - r_1)$ and $r_3 = 0$.

iv) $t_2 \neq r_2(s_1 - r_1)$ and $r_3 \neq 0$, $t_3 = 0$.

v) $t_2 \neq r_2(s_1 - r_1)$ and $r_3 t_3 \neq 0$.

See [8], in all these cases $v_n = 0$, for $n \geqslant 3$.

1. If $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ and $r_4 = 0$, from (2.3), we have

$$\langle u, Q_i(x)\rangle \neq 0, \ i = 1, 2, 3, \quad \text{and} \quad \langle u, Q_n(x)\rangle = 0, \ n \geqslant 4.$$

So, there exists a polynomial q of degree 3 such that $u = q(x)v$ [9].

Therefore,

$$Q_n(x) = P_n(x) + a_n P_{n-1}(x) + b_n P_{n-2}(x) + c_n P_{n-3}(x),$$

for each $n \geqslant 0$, with $c_n \neq 0$, $n \geqslant 3$. Again, the relation (1.1) leads to

$$s_n = a_n + r_n, \quad n \geqslant 1,$$
$$t_n = b_n + r_n a_{n-1}, \quad n \geqslant 2,$$
$$v_n = c_n + r_n b_{n-1}, \quad n \geqslant 3,$$

and $r_n c_{n-1} = 0$, $n \geqslant 4$.

So, $r_n = 0$, $n \geqslant 4$, and $v_n \neq 0$, $n \geqslant 3$. Then, this case is the degenerate 1–4 type relation.

2. If $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ and $r_4 \neq 0$, according to (2.3), we have

$$\langle u, Q_3(x)\rangle \neq 0 \quad \text{and} \quad \langle u, Q_4(x)\rangle \neq 0,$$

and if $r_n = 0$, for each $n \geqslant 5$, we get $\langle u, Q_n(x)\rangle = 0,\ n \geqslant 5$. Assuminig that exists $n \geqslant 5$ such that $r_n = 0$, putting $n_0 := \min\{n \in \mathbb{N}/n \geqslant 5,\ r_n = 0\}$, then

$$\langle u, Q_n(x)\rangle = 0,\ n \geqslant n_0, \quad \text{and} \quad \langle u, Q_n(x)\rangle \neq 0,\ 3 \leqslant n \leqslant n_0 - 1.$$

So, there exists a polynomial q of degree $n_0 - 1$ such that $u = q(x)v$ [9].

Therefore,

$$Q_n(x) = P_n(x) + \sum_{k=1}^{n_0-1} \alpha_{n,k} P_{n-k}(x),$$

where $\alpha_{n,n_0-1} \neq 0$, $n \geqslant n_0 - 1$.

Taking into account (1.1), this is not possible. Thus $r_n \neq 0$, $n \geqslant 4$.

3. If $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ and $r_4 v_4 \neq 0$, then there exists a constant c such that

$$(x - c)u = q(x)v$$

with q a polynomial of degree 3 and by (1.1), we can write for $n \geqslant 0$

$$\begin{aligned}\big\langle (x-c)u, Q_n(x)Q_{n-4}(x)\big\rangle &= \big\langle (x-c)u, (P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x) + \dots \\ &\quad + v_n P_{n-3}(x))Q_{n-4}(x)\big\rangle - r_n\big\langle (x-c)u, Q_{n-1}(x)Q_{n-4}(x)\big\rangle \\ &= v_n\big\langle u, P_{n-3}^2(x)\big\rangle - r_n\big\langle (x-c)u, Q_{n-1}(x)Q_{n-4}(x)\big\rangle.\end{aligned}$$

Consequently,

$$\begin{aligned} v_n\big\langle u, P_{n-3}^2(x)\big\rangle &= \big\langle (x-c)u, (Q_n(x) + r_n Q_{n-1}(x))Q_{n-4}(x)\big\rangle \\ = \big\langle q(x)v, (Q_n(x) + r_n Q_{n-1}(x))Q_{n-4}(x)\big\rangle &= r_n\big\langle v, q(x)Q_{n-1}(x)Q_{n-4}(x)\big\rangle = k_1 r_n\big\langle v, Q_{n-1}^2(x)\big\rangle,\end{aligned}$$

where k_1 is the leading coefficient of the polynomial q. Now, it is enough to apply (2) to obtain $r_n \neq 0$, $n \geqslant 4$, and from Definition 1.1, we have $v_n r_n \neq 0$, $n \geqslant 4$. $\rhd$

In the following proposition, we show that if $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ and $r_4 v_4 \neq 0$ this equivalence to assume that the functional $(x - c)u$ is regular.

Proposition 2.1. *Let $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ be two MOPS with respect to the regular normalized linear functionals u and v respectively, such that the relation* (1.1) *holds and the initial conditions $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ and $r_4 v_4 \neq 0$ hold. Then the following statements are equivalent:*

i) The functional $(x - c)u$ is regular.

ii) $v_n \neq r_n(t_{n-1} - r_{n-1}(s_{n-2} - r_{n-2}))$, $n \geqslant 3$.

$\lhd$ Multiplying the relation (1.1) by P_{n-1} and applying u, the same way for P_{n-2} and P_{n-3}, we get, respectively,

$$\big\langle u, Q_n(x)P_{n-1}(x)\big\rangle = (s_n - r_n)\big\langle u, P_{n-1}^2(x)\big\rangle, \quad n \geqslant 1,$$

$$\big\langle u, Q_n(x)P_{n-2}(x)\big\rangle = (t_n - r_n(s_{n-1} - r_{n-1}))\big\langle u, P_{n-2}^2(x)\big\rangle, \quad n \geqslant 2,$$

$$\big\langle u, Q_n(x)P_{n-3}(x)\big\rangle = \big(v_n - r_n(t_{n-1} - r_{n-1}(s_{n-2} - r_{n-2}))\big)\big\langle u, P_{n-3}^2(x)\big\rangle, \quad n \geqslant 3.$$

Thus

$$v_n - r_n(t_{n-1} - r_{n-1}(s_{n-2} - r_{n-2})) \neq 0 \Leftrightarrow \langle u, Q_n(x)P_{n-3}(x)\rangle \neq 0, \quad n \geqslant 3.$$

It is well known that $(x - c)u$ is regular if and only if $P_n(c) \neq 0$, for all $n \geqslant 0$ [9].

Moreover, we need to show that $\langle u, Q_{n+3}(x)P_n(x)\rangle \neq 0 \Leftrightarrow P_n(c) \neq 0$, for each $n \geqslant 0$. Either for

$$P_n(x) = \sum_{k=0}^{n} a_{nk}(x-c)^k, \quad n \geqslant 0,$$

with $a_{n0} = P_n(c)$ and $a_{nn} = 1$.

The relation between the regular functionals u and v is

$$(x-c)u = q(x)v,$$

Hence

$$\begin{gathered}\big\langle u, Q_{n+3}(x)P_n(x)\big\rangle = \big\langle (x-c)u, (x-c)^{n-1}Q_{n+3}(x)\big\rangle \\ +\sum_{k=1}^{n-1} a_{nk}\big\langle (x-c)u, (x-c)^{k-1}Q_{n+3}(x)\big\rangle + P_n(c)\langle u, Q_{n+3}(x)\rangle \\ = \big\langle v, q(x)(x-c)^{n-1}Q_{n+3}(x)\big\rangle + \sum_{k=1}^{n-1} a_{ni}\big\langle v, q(x)(x-c)^{k-1}Q_{n+3}(x)\big\rangle \\ + P_n(c)\langle u, Q_{n+3}(x)\rangle, \quad n \geqslant 0.\end{gathered}$$

Then

$$\langle u, Q_{n+3}(x)P_n(x)\rangle = P_n(c)\langle u, Q_{n+3}(x)\rangle, \quad n \geqslant 0,$$

from Lemma 2.1 and the relation (2.3), we get

$$\langle u, Q_n(x)\rangle \neq 0, \quad n \geqslant 3. \quad \triangleright$$

3. Characterization of Orthogonality

Let $\{P_n\}_{n\geqslant 0}$ be a MOPS with respect to a regular functional u and let $\{\beta_n\}_{n\geqslant 0}$, $\{\gamma_n\}_{n\geqslant 1}$ be the corresponding sequences of recurrence coefficients, so that

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geqslant 0, \tag{3.1}$$

with the initial conditions $P_0(x) = 1$, $P_{-1}(x) = 0$ and the condition $\gamma_n \neq 0$, for each $n \geqslant 1$.

In this section, we give the characterizations of the orthogonality of a sequence $\{Q_n\}_{n\geqslant 0}$ of a monic polynomials defined by a non-degenerate type relation (1.1).

From Lemma 2.1, the conditions $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ and $r_4v_4 \neq 0$ must hold, in order to have a non-degenerate 2–4 type relation with $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ MOPS and these conditions imply $v_nr_n \neq 0$, for each $n \geqslant 4$.

The following if the first characterization of the orthogonality of the sequence $\{Q_n\}_{n\geqslant 0}$.

Proposition 3.1. *Let $\{P_n\}_{n\geqslant 0}$ be a MOPS satisfies* (3.1), *and let $\{Q_n\}_{n\geqslant 0}$ be a sequence of polynomials given by the structure relation* (1.1) *with $v_3 \neq r_3(t_2 - r_2(s_1 - r_1))$ and $v_nr_n \neq 0$ for $n \geqslant 4$. Then, $\{Q_n\}_{n\geqslant 0}$ is a MOPS with recurrence coefficients $\{\widetilde{\beta}_n\}_{n\geqslant 0}$ and $\{\widetilde{\gamma}_n\}_{n\geqslant 1}$, where*

$$\widetilde{\beta}_n := \beta_n - s_{n+1} + s_n + r_{n+1} - r_n, \quad n \geqslant 0, \tag{3.2}$$

$$\begin{gathered}\widetilde{\gamma}_n := \gamma_n + t_n - t_{n+1} + s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1}) \\ - r_n(r_{n+1} - r_n - \widetilde{\beta}_n + \widetilde{\beta}_{n-1}), \quad n \geqslant 1,\end{gathered} \tag{3.3}$$

if and only if $\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3 \neq 0$, *and the following relations hold:*

$$\alpha_n s_{n-1} = s_n\gamma_{n-1} + t_n(s_{n+1} - s_n - \beta_n + \beta_{n-2}) - v_{n+1} + v_n, \quad n \geqslant 2, \tag{3.4}$$

$$\alpha_n t_{n-1} = t_n\gamma_{n-2} + v_n(s_{n+1} - s_n - \beta_n + \beta_{n-3}), \quad n \geqslant 3, \tag{3.5}$$

$$\alpha_n v_{n-1} = v_n\gamma_{n-3}, \quad n \geqslant 4, \tag{3.6}$$

$$\alpha_n r_{n-1} = r_n\widetilde{\gamma}_{n-1}, \quad n \geqslant 2, \tag{3.7}$$

where

$$\alpha_n := \gamma_n + t_n - t_{n+1} + s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1}), \quad n \geqslant 1. \tag{3.8}$$

◁ Substituting (3.1) in (1.1), for all $n \geqslant 0$, we have

$$Q_{n+1}(x) = xP_n(x) - (\beta_n - s_{n+1})P_n(x) - (\gamma_n - t_{n+1})P_{n-1}(x) + v_{n+1}P_{n-2}(x) - r_{n+1}Q_n(x). \tag{3.9}$$

Applying (1.1) to $xP_n(x)$, and substituting the recurrence relation (3.1) into (3.9) for $xP_{n-1}(x)$, $xP_{n-2}(x)$ and $xP_{n-3}(x)$, we obtain for $n \geqslant 0$

$$\begin{aligned} Q_{n+1}(x) &= (x - r_{n+1})Q_n(x) + r_n(xQ_{n-1}(x)) - (\beta_n - s_{n+1} + s_n)P_n(x) \\ &- (\gamma_n + t_n - t_{n+1} + s_n\beta_{n-1})P_{n-1}(x) - (s_n\gamma_{n-1} + t_n\beta_{n-2} + v_n - v_{n+1})P_{n-2}(x) \\ &- (v_n\beta_{n-3} + t_n\gamma_{n-2})P_{n-3}(x) - v_n\gamma_{n-3}P_{n-4}(x). \end{aligned}$$

Using relation (1.1) for P_n, and with convention $P_{-n}(x) = 0$, $n \geqslant 1$, the above relation becomes, for $n \geqslant 0$

$$\begin{aligned} Q_{n+1}(x) &= (x - \widetilde{\beta}_n)Q_n(x) - r_n(Q_n - xQ_{n-1}) - r_n(\beta_n + s_n - s_{n+1})Q_{n-1}(x) \\ &- [\gamma_n + t_n - t_{n+1} - s_n(\beta_n - \beta_{n-1} + s_n - s_{n+1})]P_{n-1}(x) \\ &- [s_n\gamma_{n-1} - t_n(\beta_n - \beta_{n-2} + s_n - s_{n+1}) + v_n - v_{n+1}]P_{n-2}(x) \\ &- [t_n\gamma_{n-2} - v_n(\beta_n - \beta_{n-3} + s_n - s_{n+1})]P_{n-3}(x) - v_n\gamma_{n-3}P_{n-4}(x), \end{aligned}$$

where $\widetilde{\beta}_n$ is given by (3.2). Then using again (1.1) for P_{n-1}, we get for $n \geqslant 0$

$$\begin{aligned} Q_{n+1}(x) &= (x - \widetilde{\beta}_n)Q_n(x) - \widetilde{\gamma}_nQ_{n-1}(x) \\ &- r_n\big[Q_n(x) - (x - \widetilde{\beta}_{n-1})Q_{n-1}(x) + \widetilde{\gamma}_{n-1}Q_{n-2}(x)\big] - (\alpha_n r_{n-1} - r_n\widetilde{\gamma}_{n-1})Q_{n-2}(x) \\ &- \big[s_n\gamma_{n-1} - t_n(\beta_n - \beta_{n-2} + s_n - s_{n+1}) + v_n - v_{n+1} - \alpha_n s_{n-1}\big]P_{n-2}(x) \\ &- \big[t_n\gamma_{n-2} - v_n(\beta_n - \beta_{n-3} + s_n - s_{n+1}) - \alpha_n t_{n-1}\big]P_{n-3}(x) \\ &- (v_n\gamma_{n-3} - \alpha_n v_{n-1})P_{n-4}(x), \end{aligned} \tag{3.10}$$

where $\widetilde{\gamma}_n$ and α_n are given by (3.3) and (3.8).

Hence from (3.10), $\{Q_n\}_{n\geqslant 0}$ be a MOPS if and only if $\widetilde{\gamma}_n \neq 0$, for $n \geqslant 1$, and the conditions (3.4)–(3.7) hold.

Suppose that $\{Q_n\}_{n\geqslant 0}$ is a MOPS, then the relation (3.10) is equivalent to

$$\begin{aligned} (f_n - \alpha_n r_{n-1})Q_{n-2}(x) &= (b_n - \alpha_n s_{n-1})P_{n-2}(x) + (c_n - \alpha_n t_{n-1})P_{n-3}(x) \\ &+ (e_n - \alpha_n v_{n-1})P_{n-4}(x), \quad n \geqslant 2, \end{aligned} \tag{3.11}$$

where

$$b_n := s_n\gamma_{n-1} + t_n(s_{n+1} - s_n - \beta_n + \beta_{n-2}) - v_{n+1} + v_n, \quad n \geqslant 2, \tag{3.12}$$

$$c_n := t_n\gamma_{n-2} + v_n(s_{n+1} - s_n - \beta_n + \beta_{n-3}), \quad n \geqslant 3, \tag{3.13}$$

$$e_n := v_n\gamma_{n-3}, \quad n \geqslant 4, \tag{3.14}$$

$$f_n := r_n\widetilde{\gamma}_{n-1}, \quad n \geqslant 2. \tag{3.15}$$

Moreover, since $v_3 \neq t_3 - r_3(s_2 - r_2)$ and $r_n \neq 0$, for $n \geqslant 4$, then by (2.3), we deduce

$$\langle u, Q_n\rangle \neq 0, \quad n \geqslant 3.$$

Applying u to both sides of (3.11), we get

$$(f_n - r_{n-1}\alpha_n)\langle u, Q_{n-2}\rangle = 0, \quad n \geqslant 5.$$

This leads to

$$f_n = r_{n-1}\alpha_n, \quad n \geqslant 5.$$

Multiplying (3.11) by P_{n-2}, P_{n-3} and P_{n-4} and applying u, we obtain for all $n \geqslant 5$

$$b_n = \alpha_n s_{n-1}, \quad c_n = \alpha_n t_{n-1}, \quad e_n = \alpha_n v_{n-1}, \quad f_n = \alpha_n r_{n-1}$$

comparing coefficients in both sides of (3.11), for $n = 2, 3$ and 4, we obtain

$$b_2 - f_2 = \alpha_2(s_1 - r_1), \tag{3.16}$$

$$b_3 - f_3 = \alpha_3(s_2 - r_2), \tag{3.17}$$

$$c_3 - b_3(s_1 - r_1) = \alpha_3(t_2 - s_2(s_1 - r_1)), \tag{3.18}$$

$$b_4 - f_4 = \alpha_4(s_3 - r_3), \tag{3.19}$$

$$c_4 - b_4(s_2 - r_2) = \alpha_4(t_3 - s_3(s_2 - r_2)), \tag{3.20}$$

$$e_4 - b_4(t_2 - r_2(s_1 - r_1)) = \alpha_4[v_3 - s_3(t_2 - r_2(s_1 - r_1))]. \tag{3.21}$$

Conversely, if (3.11) is satisfied and $\widetilde{\gamma}_n \neq 0$, for every $n \geqslant 1$, we have

$$Q_{n+1}(x) - (x - \widetilde{\beta}_n)Q_n(x) + \widetilde{\gamma}_n Q_{n-1}(x) = -r_n\left[Q_n(x) - (x - \widetilde{\beta}_{n-1})Q_{n-1}(x) + \widetilde{\gamma}_{n-1}Q_{n-2}(x)\right], \ n \geqslant 1.$$

Moreover, from (1.1), we obtain

$$Q_1(x) = P_1(x) + s_1 - r_1 = x - \beta_0 + s_1 - r_1 = x - \widetilde{\beta}_0,$$

we deduce recursively

$$Q_{n+1}(x) = (x - \widetilde{\beta}_n)Q_n(x) - \widetilde{\gamma}_n Q_{n-1}(x), \quad n \geqslant 1.$$

Thus, $\{Q_n\}_{n\geqslant 0}$ is a MOPS with recurrence coefficients $\{\widetilde{\beta}_n\}_{n\geqslant 0}$ and $\{\widetilde{\gamma}_n\}_{n\geqslant 1}$. $\rhd$

Now, we show that the orthogonality of the sequence $\{Q_n\}_{n\geqslant 0}$ can be also characterized by the fact that there are four sequences depending on the parameters r_n, s_n, t_n, v_n and the recurrence coefficients which remain constants.

Theorem 3.1. *Let $\{P_n\}_{n\geqslant 0}$ be a MOPS and let $\{Q_n\}_{n\geqslant 0}$ be a sequence of polynomials given by* (1.1). *Then the following statements are equivalent:*

(*i*) *$\{Q_n\}_{n\geqslant 0}$ is a MOPS with recurrence coefficients $\{\widetilde{\beta}_n\}_{n\geqslant 0}$ and $\{\widetilde{\gamma}_n\}_{n\geqslant 1}$ given by* (3.2) *and* (3.3).

(ii) It holds $\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3 \neq 0$ *together with initial conditions* (3.16)–(3.21) *and*

$$v_5\gamma_2 = v_4(\gamma_5 + t_5 - t_6 + s_5(s_6 - s_5 - \beta_5 + \beta_4)) \tag{3.22}$$

and there exist four complex numbers A, B, C *and* D *such that, for each* $n \geqslant 4$

$$A = \widetilde{\beta}_n - r_{n+1} - \frac{\widetilde{\gamma}_n}{r_n}, \tag{3.23}$$

$$B = \frac{t_n}{v_n}\gamma_{n-2} + s_{n+1} - \beta_n - \beta_{n-1} - \beta_{n-2}, \tag{3.24}$$

$$\begin{aligned} C = \frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2} + \Big(\frac{t_n}{v_n}\gamma_{n-2} + s_{n+1} - \beta_{n-2}\Big)(s_{n+1} - \beta_n - \beta_{n-1}) \\ - s_{n+1}(s_{n+2} - \beta_{n+1}) + \beta_{n-1}\beta_n - \gamma_{n+1} - \gamma_n - \gamma_{n-1} + t_{n+2}, \end{aligned} \tag{3.25}$$

$$\begin{aligned} D = \frac{\gamma_n\gamma_{n-1}\gamma_{n-2}}{v_n} + \frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2}(s_{n+1} - \beta_n) \\ + \frac{t_n}{v_n}\gamma_{n-2}\big[s_{n+1}(s_{n+1} - s_{n+2} + \beta_{n+1} - \beta_n - \beta_{n-1}) + \beta_n\beta_{n-1} - \gamma_{n+1} - \gamma_n + t_{n+2}\big] \\ + (s_{n+2} - s_{n+1} - \beta_{n+1} + \beta_{n-1} + \beta_{n-2})(\alpha_{n+1} - t_{n+1}) \\ + (s_{n+1} - \beta_n)(\beta_{n-1}\beta_{n-2} - \gamma_{n-1}) - (s_{n+1} - \beta_{n-2})\gamma_n + v_{n+2}. \end{aligned} \tag{3.26}$$

$\lhd$ Observe that the conditions (3.4)–(3.7) in the Proposition 3.1 may be written as, for each $n \geqslant 5$

$$s_{n-1}\frac{v_n}{v_{n-1}}\gamma_{n-3} = s_n\gamma_{n-1} + t_n(s_{n+1} - s_n - \beta_n + \beta_{n-2}) - v_{n+1} + v_n, \tag{3.27}$$

$$t_{n-1}\frac{v_n}{v_{n-1}}\gamma_{n-3} = t_n\gamma_{n-2} + v_n(s_{n+1} - s_n - \beta_n + \beta_{n-3}), \tag{3.28}$$

$$\frac{v_n}{v_{n-1}}\gamma_{n-3} = \gamma_n + t_n - t_{n+1} + s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1}), \tag{3.29}$$

$$\frac{r_n}{r_{n-1}}\widetilde{\gamma}_{n-1} = \widetilde{\gamma}_n + r_n(r_{n+1} - r_n - \widetilde{\beta}_n + \widetilde{\beta}_{n-1}), \tag{3.30}$$

moreover $\{Q_n\}_{n\geqslant 0}$ is a MOPS if and only if the conditions $\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3 \neq 0$, the initial conditions (3.16)–(3.21) and the above equations (3.27), (3.28), (3.29) and (3.30) hold.

Firstly, we show that (3.27)–(3.30) $\Rightarrow$ (3.22)–(3.26).

For $n = 5$ in (3.29), yields (3.22).

From (3.30), dividing the left and the right hand sides by r_n, we get

$$\widetilde{\beta}_n - r_{n+1} - \frac{\widetilde{\gamma}_n}{r_n} = \widetilde{\beta}_{n-1} - r_n - \frac{\widetilde{\gamma}_{n-1}}{r_{n-1}}, \quad n \geqslant 5. \tag{3.31}$$

Hence (3.23) holds.

Now, we will deduce (3.24).

Using (3.28), dividing the left and the right hand sides by v_n, we obtain

$$\frac{t_n}{v_n}\gamma_{n-2} + s_{n+1} - \beta_n - \beta_{n-1} - \beta_{n-2} = \frac{t_{n-1}}{v_{n-1}}\gamma_{n-3} + s_n - \beta_{n-1} - \beta_{n-2} - \beta_{n-3}, \quad n \geqslant 5. \tag{3.32}$$

Hence (3.24) holds.

Next, we will deduce (3.25).

From (3.27), multiplying the left and the right hand sides by $\frac{\gamma_{n-2}}{v_n}$, we obtain

$$\begin{aligned}&\frac{s_{n-1}}{v_{n-1}}\gamma_{n-2}\gamma_{n-3}+\frac{t_n}{v_n}\gamma_{n-2}(s_n-\beta_{n-1}-\beta_{n-2})\\&=\frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2}+\frac{t_n}{v_n}\gamma_{n-2}(s_{n+1}-\beta_n-\beta_{n-1})+\Big(1-\frac{v_{n+1}}{v_n}\Big)\gamma_{n-2},\end{aligned}\tag{3.33}$$

taking into account (3.29) for $n+1$ instead of n, for each $n\geqslant 5$, we have

$$\begin{aligned}\frac{s_{n-1}}{v_{n-1}}\gamma_{n-2}\gamma_{n-3}+\frac{t_n}{v_n}\gamma_{n-2}(s_n-\beta_{n-1}-\beta_{n-2})&=\frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2}+\frac{t_n}{v_n}\gamma_{n-2}(s_{n+1}-\beta_n-\beta_{n-1})\\&+\gamma_{n-2}-\gamma_{n+1}-t_{n+1}+t_{n+2}-s_{n+1}(s_{n+2}-s_{n+1}-\beta_{n+1}+\beta_n),\end{aligned}$$

using (3.24), in the above expression, for each $n\geqslant 5$, we get

$$\begin{aligned}&\frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2}+\Big(\frac{t_n}{v_n}\gamma_{n-2}+s_{n+1}-\beta_{n-1}-\beta_{n-2}\Big)(s_{n+1}-\beta_n-\beta_{n-1})\\&\quad-\gamma_{n+1}-\gamma_n-\gamma_{n-1}+t_{n+2}-s_{n+1}(s_{n+2}-\beta_{n+1}-\beta_{n-1})-\beta_{n-1}^2\\&=\frac{s_{n-1}}{v_{n-1}}\gamma_{n-2}\gamma_{n-3}+\Big(\frac{t_{n-1}}{v_{n-1}}\gamma_{n-3}+s_n-\beta_{n-2}-\beta_{n-3}\Big)(s_n-\beta_{n-1}-\beta_{n-2})\\&\quad-\gamma_n-\gamma_{n-1}-\gamma_{n-2}+t_{n+1}-s_n(s_{n+1}-\beta_n-\beta_{n-2})-\beta_{n-2}^2.\end{aligned}\tag{3.34}$$

Lastly, we will deduce (3.26).

From (3.29), multiplying the left and the right hand sides by $\dfrac{\gamma_{n-1}\gamma_{n-2}}{v_n}$, we obtain

$$\begin{aligned}&\frac{\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}}{v_{n-1}}+\frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2}(s_n-\beta_{n-1})\\&=\frac{\gamma_n\gamma_{n-1}\gamma_{n-2}}{v_n}+\frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2}(s_{n+1}-\beta_n)+\Big(\frac{t_n}{v_n}-\frac{t_{n+1}}{v_n}\Big)\gamma_{n-1}\gamma_{n-2},\end{aligned}$$

taking into account (3.28) for $n+1$ instead of n, for each $n\geqslant 5$, we have

$$\begin{aligned}\frac{\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}}{v_{n-1}}+\frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2}(s_n-\beta_{n-1})&=\frac{\gamma_n\gamma_{n-1}\gamma_{n-2}}{v_n}+\frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2}(s_{n+1}-\beta_n)\\&+\frac{t_n}{v_n}\gamma_{n-2}\big(\gamma_{n-1}-\gamma_{n+1}-t_{n+1}+t_{n+2}-s_{n+1}(s_{n+2}-s_{n+1}-\beta_{n+1}+\beta_n)\big)\\&+\big(s_{n+2}-s_{n+1}-\beta_{n+1}+\beta_{n-2}\big)\big(\gamma_{n+1}+t_{n+1}-t_{n+2}+s_{n+1}(s_{n+2}-s_{n+1}-\beta_{n+1}+\beta_n)\big),\end{aligned}\tag{3.35}$$

using (3.33) and (3.1), we obtain

$$\begin{aligned}&\frac{\gamma_n\gamma_{n-1}\gamma_{n-2}}{v_n}+\Big(\frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2}-\gamma_{n-1}\Big)(s_{n+1}-\beta_n)\\&+\frac{t_n}{v_n}\gamma_{n-2}\big(-\gamma_n-\gamma_{n+1}+t_{n+2}-s_{n+1}(s_{n+2}-s_{n+1}-\beta_{n+1}+\beta_n+\beta_{n-1})+\beta_n\beta_{n-1}\big)\\&+\big(s_{n+2}-s_{n+1}-\beta_{n+1}+\beta_{n-2}-(s_n-\beta_{n-1})\big)\big(\gamma_{n+1}+t_{n+1}-t_{n+2}+s_{n+1}(s_{n+2}-s_{n+1}\\&-\beta_{n+1}+\beta_n)\big)=\frac{\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}}{v_{n-1}}+\Big(\frac{s_{n-1}}{v_{n-1}}\gamma_{n-2}\gamma_{n-3}-\gamma_{n-2}\Big)(s_n-\beta_{n-1})-(s_n-\beta_{n-3})\gamma_{n-1}\\&+\frac{t_{n-1}}{v_{n-1}}\gamma_{n-3}\big(-\gamma_{n-1}-\gamma_n+t_{n+1}-s_n(s_{n+1}-s_n-\beta_n+\beta_{n-1}+\beta_{n-2})+\beta_{n-1}\beta_{n-2}\big)\\&+\big(-s_{n+1}+s_n+\beta_n-\beta_{n-3}\big)\big[-\gamma_n+t_{n+1}-s_n(s_{n+1}-s_n-\beta_n+\beta_{n-1}+\beta_{n-2})+\beta_{n-1}\beta_{n-2}\big],\end{aligned}$$

taking into account (3.8) and (3.12) in the above expressions and by straightforward computation, for each $n \geqslant 5$, we get

$$\begin{aligned}
&\frac{\gamma_n\gamma_{n-1}\gamma_{n-2}}{v_n} + \frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2}(s_{n+1}-\beta_n) \\
&+\frac{t_n}{v_n}\gamma_{n-2}\big[s_{n+1}(s_{n+1}-s_{n+2}+\beta_{n+1}-\beta_n-\beta_{n-1})+\beta_n\beta_{n-1}-\gamma_{n+1}-\gamma_n+t_{n+2}\big] \\
&+(s_{n+2}-s_{n+1}-\beta_{n+1}+\beta_{n-1}+\beta_{n-2})(\alpha_{n+1}-t_{n+1})+(s_{n+1}-\beta_n)(\beta_{n-1}\beta_{n-2}-\gamma_{n-1}) \\
&-(s_{n+1}-\beta_{n-2})\gamma_n + v_{n+2} = \frac{\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}}{v_{n-1}} + \frac{s_{n-1}}{v_{n-1}}\gamma_{n-2}\gamma_{n-3}(s_n-\beta_{n-1}) \\
&+\frac{t_{n-1}}{v_{n-1}}\gamma_{n-3}\big[s_n(s_n-s_{n+1}+\beta_n-\beta_{n-1}-\beta_{n-2})+\beta_{n-1}\beta_{n-2}-\gamma_n-\gamma_{n-1}+t_{n+1}\big] \\
&+(s_{n+1}-s_n-\beta_n+\beta_{n-2}+\beta_{n-3})(\alpha_n-t_n)+(s_n-\beta_{n-1})(\beta_{n-2}\beta_{n-3}-\gamma_{n-2}) \\
&-(s_n-\beta_{n-3})\gamma_{n-1}+v_{n+1}.
\end{aligned} \tag{3.36}$$

Secondly, we show that (3.23)–(3.26) $\Rightarrow$ (3.27)–(3.30).

Notice that, the relations (3.27)–(3.30) are equivalent to (3.4)–(3.7) and the relations (3.23)–(3.26) are equivalent to (1.1)–(3.36) then, it is enough to show (1.1)–(3.36) $\Rightarrow$ (3.27)–(3.30).

From (1.1), multiplying the left and the right hand sides by r_n, yields (3.30).

From (3.1), multiplying the left and the right hand sides by v_n, yields (3.28).

From (3.35), we have

$$\begin{aligned}
&\frac{\gamma_{n-1}\gamma_{n-2}}{v_n}\left[\frac{v_n}{v_{n-1}}\gamma_{n-3}-\gamma_n-t_n+t_{n+1}+s_n(s_n-s_{n+1}-\beta_{n-1}+\beta_n)\right] \\
&=\frac{\gamma_{n-2}}{v_n}\big[-t_n\alpha_{n+1}+t_{n+1}\gamma_{n-1}+v_{n+1}(s_{n+2}-s_{n+1}-\beta_{n+1}+\beta_{n-2})\big] \\
&+\big(s_{n+2}-s_{n+1}-\beta_{n+1}+\beta_{n-2}\big)\Big(-\frac{v_{n+1}}{v_n}\gamma_{n-2}+\gamma_{n+1}+t_{n+1}-t_{n+2} \\
&\qquad + s_{n+1}(s_{n+2}-s_{n+1}-\beta_{n+1}+\beta_n)\Big),
\end{aligned}$$

using (3.5), for $n+1$ instead of n, we get

$$\begin{aligned}
&\frac{\gamma_{n-1}\gamma_{n-2}}{v_n}\left[\frac{v_n}{v_{n-1}}\gamma_{n-3}-\gamma_n-t_n+t_{n+1}+s_n(s_n-s_{n+1}-\beta_{n-1}+\beta_n)\right] \\
&=(s_{n+1}-s_{n+2}-\beta_{n-2}+\beta_{n+1})\Big[\frac{v_{n+1}}{v_n}\gamma_{n-2}-\gamma_{n+1}-t_{n+1}+t_{n+2} \\
&\qquad + s_{n+1}(s_{n+1}-s_{n+2}-\beta_n+\beta_{n+1})\Big],
\end{aligned}$$

according to (3.22) and as v_4 not equal to 0, we have

$$\frac{v_5}{v_4}\gamma_2-\gamma_5-t_5+t_6+s_5(s_5-s_6-\beta_4+\beta_5)=0,$$

then

$$\frac{v_n}{v_{n-1}}\gamma_{n-3}-\gamma_n-t_n+t_{n+1}+s_n(s_n-s_{n+1}-\beta_{n-1}+\beta_n)=0, \quad n\geqslant 5.$$

Hence (3.29) holds.

Taking into account the expression of $\frac{t_{n-1}}{v_{n-1}}\gamma_{n-3}$, obtained from (3.1), the relation (3.34) rewrites as

$$\begin{aligned}&\frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2}+\frac{t_n}{v_n}\gamma_{n-2}(s_{n+1}-\beta_n-\beta_{n-1})-\gamma_{n+1}-\gamma_n-\gamma_{n-1}+t_{n+2}\\&-s_{n+1}(s_{n+2}-s_{n+1}-\beta_{n+1}+\beta_n)+(\beta_n+\beta_{n-1}-s_{n+1})(\beta_{n-1}+\beta_{n-2})-\beta_{n-1}^2\\&=\frac{s_{n-1}}{v_{n-1}}\gamma_{n-2}\gamma_{n-3}+\Big(\frac{t_n}{v_n}\gamma_{n-2}+s_{n+1}-s_n-\beta_n+\beta_{n-3}\Big)(s_n-\beta_{n-1}-\beta_{n-2})\\&\qquad-\gamma_n-\gamma_{n-1}-\gamma_{n-2}+t_{n+1}-s_n(s_{n+1}-s_n-\beta_n+\beta_{n-1})\\&\qquad+(\beta_{n-1}+\beta_{n-2}-s_n)(\beta_{n-2}+\beta_{n-3})-\beta_{n-2}^2,\end{aligned}$$

hence

$$\begin{aligned}&\frac{s_{n-1}}{v_{n-1}}\gamma_{n-2}\gamma_{n-3}=\frac{s_n}{v_n}\gamma_{n-1}\gamma_{n-2}+\frac{t_n}{v_n}\gamma_{n-2}(s_{n+1}-s_n-\beta_n+\beta_{n-2})\\&+\gamma_{n-2}-(\gamma_{n+1}+t_{n+1}-t_{n+2}+s_{n+1}(s_{n+2}-s_{n+1}-\beta_{n+1}+\beta_n)),\end{aligned}$$

using (3.29) for $n+1$ instead of n and simplifying, yield (3.27). $\triangleright$

In the following theorem, we observe that there is a relation between regular linear functionals when $\{Q_n\}_{n\geqslant 0}$ is a MOPS with respect to a regular linear functional v.

Theorem 3.2. *Let $\{P_n\}_{n\geqslant 0}$ be a MOPS with respect to a regular linear functional u and the sequence of monic polynomials $\{Q_n\}_{n\geqslant 0}$ be given by the relation* (1.1). *If $\{Q_n\}_{n\geqslant 0}$ is a MOPS with respect to a regular linear functional v, then*

$$k(x-c)u=(x^3+ax^2+bx+d)v \tag{3.37}$$

with $c,a,b,d\in\mathbb{C}$ and $k\in\mathbb{C}\setminus\{0\}$ and the normalizations for these linear functionals $\langle u,1\rangle=\langle v,1\rangle=1$.

$\triangleleft$ Applying the regular linear functional u corresponding to the MOPS $\{P_n\}_{n\geqslant 0}$ in (1.1), we obtain, for each $n\geqslant 4$

$$\langle (x-c)u, Q_n(x)\rangle = 0.$$

Then, according to [2] taking into account the relation (1.1), we expand the linear functional u in terms of the dual basis $\big\{\frac{Q_i v}{\langle v,Q_i^2\rangle}\big\}_{i\geqslant 0}$ of the MOPS $\{Q_n\}_{n\geqslant 0}$ as

$$(x-c)u=\sum_{i=0}^{3}\frac{\langle (x-c)u,Q_i\rangle}{\langle v,Q_i^2\rangle}Q_i v.$$

Since $\{Q_n\}_{n\geqslant 0}$ is a $MOPS$ with respect to v, the recurrence coefficients $\{\widetilde{\beta}_n\}_{n\geqslant 0}$ and $\{\widetilde{\gamma}_n\}_{n\geqslant 1}$ are given by (3.2) and (3.3). Moreover

$$\widetilde{\gamma}_n=\frac{\langle v,Q_n^2\rangle}{\langle v,Q_{n-1}^2\rangle}\neq 0,\quad n\geqslant 1. \tag{3.38}$$

Indeed, making both sides of (3.37) acting on the polynomials Q_0, Q_1, Q_2 and Q_3, and taking into account (2.1), we get

$$k(\beta_0-c)=\widetilde{\beta}_0^3+\big(2\widetilde{\beta}_0+\widetilde{\beta}_1\big)\widetilde{\gamma}_1+\big(\widetilde{\gamma}_1+\widetilde{\beta}_0^2\big)a+\widetilde{\beta}_0 b+d, \tag{3.39}$$

$$k\big[\gamma_1+(\beta_0-c)(s_1-r_1)\big]=\big(\widetilde{\beta}_0^2+\widetilde{\beta}_1^2+\widetilde{\beta}_0\widetilde{\beta}_1+\widetilde{\gamma}_1+\widetilde{\gamma}_2\big)\widetilde{\gamma}_1+a\big(\widetilde{\beta}_0+\widetilde{\beta}_1\big)\widetilde{\gamma}_1+\widetilde{\gamma}_1 b, \tag{3.40}$$

$$k\big\{\gamma_1(s_2-r_2)+(\beta_0-c)[t_2-r_2(s_1-r_1)]\big\}=\big(\widetilde{\beta}_0+\widetilde{\beta}_1+\widetilde{\beta}_2+a\big)\widetilde{\gamma}_1\widetilde{\gamma}_2, \tag{3.41}$$

$$k\big\{\gamma_1[t_3-r_3(s_2-r_2)]+(\beta_0-c)[v_3-r_3(t_2-r_2(s_1-r_1))]\big\}=\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3, \tag{3.42}$$

where $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, $\widetilde{\gamma}_3$ are given by (3.2) and (3.3).

Using the relations (3.39)–(3.42) and taking into account (2.2), thus, the values of c, a, b, d and k are given as follows

$$\begin{aligned}
k&=\frac{r_4}{v_4}\,\frac{\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3}{\gamma_1},\quad c=\beta_0-\frac{\gamma_1}{r_4}\,\frac{v_4-r_4(t_3-r_3(s_2-r_2))}{v_3-r_3(t_2-r_2(s_1-r_1))},\\
a&=-\widetilde{\beta}_0-\widetilde{\beta}_1-\widetilde{\beta}_2+\frac{\widetilde{\gamma}_3}{v_4}\,\frac{r_4v_3(s_2-r_2)+(v_4-r_4t_3)[t_2-r_2(s_1-r_1)]}{v_3-r_3(t_2-r_2(s_1-r_1))},\\
b&=\widetilde{\beta}_0\widetilde{\beta}_1+\widetilde{\beta}_0\widetilde{\beta}_2+\widetilde{\beta}_1\widetilde{\beta}_2-\widetilde{\gamma}_1-\widetilde{\gamma}_2-\frac{(\widetilde{\beta}_0+\widetilde{\beta}_1)\widetilde{\gamma}_3}{v_4}\,\frac{r_4v_3(s_2-r_2)+(v_4-r_4t_3)[t_2-r_2(s_1-r_1)]}{v_3-r_3(t_2-r_2(s_1-r_1))}\\
&\quad+\frac{\widetilde{\gamma}_2\widetilde{\gamma}_3}{v_4}\,\frac{r_4(v_3-r_3t_2)+(v_4-r_4(t_3-r_3s_2))(s_1-r_1)}{v_3-r_3(t_2-r_2(s_1-r_1))},\\
d&=-\widetilde{\beta}_0\widetilde{\beta}_1\widetilde{\beta}_2+\widetilde{\beta}_0\widetilde{\gamma}_2+\widetilde{\beta}_2\widetilde{\gamma}_1-\widetilde{\beta}_0\,\frac{\widetilde{\gamma}_2\widetilde{\gamma}_3}{v_4}\,\frac{r_4(v_3-r_3t_2)+(v_4-r_4(t_3-r_3s_2))(s_1-r_1)}{v_3-r_3(t_2-r_2(s_1-r_1))}\\
&\quad+\frac{(\widetilde{\beta}_0\widetilde{\beta}_1-\widetilde{\gamma}_1)\widetilde{\gamma}_3}{v_4}\,\frac{r_4v_3(s_2-r_2)+(v_4-r_4t_3)[t_2-r_2(s_1-r_1)]}{v_3-r_3(t_2-r_2(s_1-r_1))}\\
&\quad+\frac{\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3}{v_4}\,\frac{v_4-r_4(t_3-r_3(s_2-r_2))}{v_3-r_3(t_2-r_2(s_1-r_1))}.\quad\triangleright
\end{aligned}$$

REMARK 3.1. The constants A, B, C and D appearing in the Theorem 3.2 are, respectively, the coefficients c, a, b and d of the polynomial which relate the two regular linear functionals.

4. A Particular Case

In this section, we will discuss a special case of relation (1.1).

Let us consider the symmetric MOPS $\{P_n\}_{n\geqslant 1}$, this means that $\beta_n=0$, for each $n\geqslant 0$. From Proposition 3.1, the equations (3.4), (3.5), (3.6) and (3.7) become, for each $n\geqslant 5$

$$s_{n+1}=s_n+\frac{t_{n-1}}{v_{n-1}}\gamma_{n-3}-\frac{t_n}{v_n}\gamma_{n-2}, \tag{4.1}$$

$$t_{n+1}=t_n+\gamma_n-\frac{v_n}{v_{n-1}}\gamma_{n-3}+s_n(s_{n+1}-s_n), \tag{4.2}$$

$$v_{n+1}=v_n+s_n\gamma_{n-1}+t_n(s_{n+1}-s_n)-s_{n-1}\big[\gamma_n+t_n-t_{n+1}+s_n(s_{n+1}-s_n)\big], \tag{4.3}$$

$$r_{n+1}=r_n+\frac{v_n}{v_{n-1}}\,\frac{\gamma_{n-3}}{r_n}-\frac{v_{n+1}}{v_n}\,\frac{\gamma_{n-2}}{r_{n+1}},\quad n\geqslant 4. \tag{4.4}$$

The equations (3.2) and (3.3) become

$$\begin{aligned}
\widetilde{\beta}_n&=s_n-s_{n+1}+r_{n+1}-r_n,\quad n\geqslant 0,\\
\widetilde{\gamma}_n&=\gamma_n+t_n-t_{n+1}+s_n(s_{n+1}-s_n)-r_n(s_{n+1}-s_n+\widetilde{\beta}_{n-1}),\quad n\geqslant 1,
\end{aligned}$$

for each $n \geqslant 5$, we have

$$\begin{aligned}\widetilde{\beta}_n &= \frac{t_n}{v_n}\gamma_{n-2} - \frac{t_{n-1}}{v_{n-1}}\gamma_{n-3} + \frac{v_n}{v_{n-1}}\frac{\gamma_{n-3}}{r_n} - \frac{v_{n+1}}{v_n}\frac{\gamma_{n-2}}{r_{n+1}} \\ &= \frac{\gamma_{n-2}}{v_n}\left(t_n - \frac{v_{n+1}}{r_{n+1}}\right) - \frac{\gamma_{n-3}}{v_{n-1}}\left(t_{n-1} - \frac{v_n}{r_n}\right),\end{aligned} \tag{4.5}$$

$$\widetilde{\gamma}_n = \frac{v_n}{v_{n-1}}\gamma_{n-3} - r_n\left(\frac{t_{n-1}}{v_{n-1}}\gamma_{n-3} - \frac{t_n}{v_n}\gamma_{n-2}\right) - r_n\widetilde{\beta}_{n-1}. \tag{4.6}$$

In this case, we treat the following three subcases.

i) If $s_n = s_1$ and $r_{n-1} = r_1$, for each $n \geqslant 5$, from (4.1) and (4.4), we obtain

$$\begin{aligned}\frac{t_n}{v_n}\gamma_{n-2} &= \frac{t_{n-1}}{v_{n-1}}\gamma_{n-3} = \ldots = \frac{t_3}{v_3}\gamma_1, \\ \frac{v_{n+1}}{v_n}\gamma_{n-2} &= \frac{v_n}{v_{n-1}}\gamma_{n-3} = \ldots = \frac{v_4}{v_3}\gamma_1,\end{aligned}$$

the relation (4.6) yields

$$\widetilde{\gamma}_n = \frac{v_4}{v_3}\gamma_1, \quad n \geqslant 5. \tag{4.7}$$

We conclude that $\widetilde{\beta}_n = 0$ and $\widetilde{\gamma}_n$ are constants, for each $n \geqslant 5$.

From (4.2), we have

$$t_{n+1} = t_n + \gamma_n - \frac{v_4}{v_3}\gamma_1. \tag{4.8}$$

ii) If $s_n = s_1$, $r_{n-1} = r_1$ and $t_n = t_2$, for each $n \geqslant 5$, from (4.8) and (4.7), we get

$$\widetilde{\gamma}_n = \gamma_n, \quad n \geqslant 5.$$

The coefficients γ_n are constants, for each $n \geqslant 5$, then $\{P_n\}_{n\geqslant 0}$ is the sequence of anti-associated polynomials of order 5 for the Chebyshev polynomials of the second kind [10].

iii) If $r_{n-1} = r_1$, $s_n = s_1$, $t_n = t_2$ and $v_n = v_3$, for each $n \geqslant 5$, from (4.3), we have

$$v_{n+1} = v_n + s_n\gamma_{n-1} - s_{n-1}\gamma_n, \quad n \geqslant 5,$$

hence

$$v_{n+1} = v_n + s_1(\gamma_{n-1} - \gamma_n), \quad n \geqslant 6,$$

it is clear that $s_1(\gamma_{n-1} - \gamma_n) = 0$, for all $n \geqslant 6$.

REMARK 4.1. If $r_{n-1} = r_1$, $s_n = s_1$ and $t_n = t_2$ or $r_{n-1} = r_1$, $s_n = s_1$, $t_n = t_2$ and $v_n = v_3$, for each $n \geqslant 5$, then

$$\begin{aligned}\widetilde{\beta}_n &= 0, \quad n \geqslant 5, \\ \widetilde{\gamma}_n &= \gamma_n = \gamma_5, \quad n \geqslant 5.\end{aligned}$$

EXAMPLE 4.1. Let $\{P_n\}_{n\geqslant 0}$ be the sequence of monic Chebyshev polynomials of the second kind orthogonal with respect to the weight function $\mathscr{W}(x) = (1-x^2)^{1/2}$ on $(-1, 1)$. Then $\beta_n = 0$, $n \geqslant 0$, $\gamma_n = \frac{1}{4}$, $n \geqslant 1$, and the relations (4.1), (4.2), (4.3) and (4.4), for each $n \geqslant 5$, become

$$s_{n+1} = s_n + \frac{1}{4}\left(\frac{t_{n-1}}{v_{n-1}} - \frac{t_n}{v_n}\right), \quad t_{n+1} = t_n + \frac{1}{4}\left(1 - \frac{v_n}{v_{n-1}}\right) + s_n(s_{n+1} - s_n),$$

$$v_{n+1} = v_n + \frac{1}{4}s_n + t_n(s_{n+1} - s_n) - s_{n-1}\left[\frac{1}{4} + t_n - t_{n+1} + s_n(s_{n+1} - s_n)\right],$$

$$r_n = r_{n-1} + \frac{1}{4}\left(\frac{v_{n-1}}{r_{n-1}v_{n-2}} - \frac{v_n}{r_n v_{n-1}}\right).$$

Assume that $r_{n-1} = r_1$, $s_n = s_1$ and $t_n = t_2$, for each $n \geqslant 5$, we obtain

$$v_{n+1} = v_n + \frac{1}{4}(s_n - s_{n-1}), \quad n \geqslant 5,$$

in particular,

$$v_{n+1} = v_n, \quad n \geqslant 6.$$

In this situation, we deduce constant connection coefficients, for $n \geqslant 6$.

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О НОВОЙ КОМБИНАЦИИ ПОСЛЕДОВАТЕЛЬНОСТИ ОРТОГОНАЛЬНЫХ ПОЛИНОМОВ

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Аннотация. В настоящая статья посвящена следующей обратной задаче. Для последовательность полиномов от одной переменной $\{P_n\}_{n\geqslant 0}$, ортогональных относительно квазиопределенного линейного функционала u, выяснить условия существования последовательности ортогональных полиномов $\{Q_n\}_{n\geqslant 0}$, для которых имеет место разложение $Q_n(x) + r_nQ_{n-1}(x) = P_n(x) + s_nP_{n-1}(x) + t_nP_{n-2}(x) + v_nP_{n-3}(x)$, $n \geqslant 0$, где $v_nr_n \neq 0$, для всех $n \geqslant 4$. Показано, что ортогональность последовательности $\{Q_n\}_{n\geqslant 0}$ характеризуется существованием последовательностей, зависящих от параметров r_n, s_n, t_n, v_n и постоянных рекуррентных коэффициентов. Кроме того, установлено, что соотношение между соответствующими линейными функционалами имеет вид $k(x-c)u = (x^3 + ax^2 + bx + d)v$, где $c, a, b, d \in \mathbb{C}$ and $k \in \mathbb{C} \setminus \{0\}$. Рассмотрены также подклассы для которых параметры r_n, s_n, t_n и v_n легко вычисляются. В конце приводятся иллюстрирующие примеры.

Ключевые слова: ортогональный полином, линейный функционал, обратная задача, полиномы Чебышева.

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