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ON FINITE HOMOGENEOUS METRIC SPACES#

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Abstract. This survey is devoted to recently obtained results on finite homogeneous metric spaces. The main subject of discussion is the classification of regular and semiregular polytopes in Euclidean spaces by whether or not their vertex sets have the normal homogeneity property or the Clifford — Wolf homogeneity property. Every finite homogeneous metric subspace of an Euclidean space represents the vertex set of a compact convex polytope with the isometry group that is transitive on the set of vertices, moreover, all these vertices lie on some sphere. Consequently, the study of such subsets is closely related to the theory of convex polytopes in Euclidean spaces. The normal generalized homogeneity and the Clifford — Wolf homogeneity describe more stronger properties than the homogeneity. Therefore, it is natural to first check the presence of these properties for the vertex sets of regular and semiregular polytopes. In addition to the classification results, the paper contains a description of the main tools for the study of the relevant objects.

Key words: Archimedean solid, finite Clifford — Wolf homogeneous metric space, finite homogeneous metric space, finite normal homogeneous metric space, Gosset polytope, Platonic solid, regular polytope, semiregular polytope.

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Introduction

In [1], the authors introduced and considered the class of finite homogeneous metric spaces, its subclasses of (generalized) normal homogeneous spaces and Clifford — Wolf homogeneous spaces, as well as relationships between these classes. Similar classes were studied for Riemannian manifolds in [2–6].

It was given the description of the classes under consideration in terms of graph theory. This description allows to construct some particular examples of finite metric spaces with unusual properties. For instance, the Kneser graphs are fruitful sources of such quite unexpected examples [1].

A finite homogeneous metric subspace of an Euclidean space represents the vertex set of a compact convex polytope with the isometry group that is transitive on the vertex set;

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in each case, all vertices lie on a sphere. In [1, 7, 8], the authors obtained the complete description of the metric properties of the vertex sets of regular and semiregular polytopes in Euclidean spaces from the point of view of the normal homogeneity and the Clifford — Wolf homogeneity. In this survey, we discuss the corresponding classification along with other important properties of finite homogeneous metric spaces, in particular, homogeneous polytopes in Euclidean spaces.

The paper is organized as follows. In Section 1 we consider general properties of the class of homogeneous metric spaces and its important subclasses. Section 2 is devoted to some special properties of the class of finite homogeneous metric spaces. In Section 3 we discuss some properties of finite homogeneous subspaces of Euclidean spaces. The most important results on regular and semiregular polytopes in Euclidean spaces are discussed in Section 4. Finally, in Section 5 we consider the classification of regular and semiregular polytopes in Euclidean spaces whose vertex sets have the normal homogeneity property or the Clifford — Wolf homogeneity property.

1. General Metric Spaces

For a given metric space (M,d), we denote by $\mathrm{Isom}\,(M,d)$ its isometry group.

Definition 1. A metric space (M,d) is called *homogeneous*, if for every $x,y \in M$ there exists an isometry of (M,d), moving x to y, i. e. the isometry group $\mathrm{Isom}\,(M,d)$ acts transitively on M.

It should be noted that some proper subgroups of $\mathrm{Isom}\,(M,d)$ could be also act transitively on M. For example, the isometry group of the sphere S^{2m-1} with the metric, induced by the Euclidean metric of $\mathbb{R}^{2m}$, is the orthogonal group $O(2m)$, that acts transitively. On the other hand, the special orthogonal group $SO(2m)$, the unitary group $U(m)$, and the special unitary group $SU(m)$ also act transitively on S^{2m-1}.

Definition 2. Let (M,d) be a metric space and $x \in M$. An isometry $f : M \to M$ is called *a $\delta(x)$-translation* or *a δ-translation at the point x*, if x is a point of maximal displacement of f, i. e. for every $y \in M$ the relation $d(y,f(y)) \leqslant d(x,f(x))$ holds.

Definition 3. Let (M,d) be a metric space. An isometry $f : M \to M$ is called *a Clifford — Wolf translation* (*CW-translation*), if f moves all points of (M,d) the same distance, i. e. $d(y,f(y)) = d(x,f(x))$ for every $x,y \in M$.

Let us recall one well known fact, which gives us an useful technical tool.

Proposition 1. *Let (M,d) be a metric space. If $f \in \mathrm{Isom}\,(M,d)$ is such that the group $G = \{g \in \mathrm{Isom}\,(M,d) \,|\, gf = fg\}$, i. e. the centralizer of f in $\mathrm{Isom}\,(M,d)$, acts transitively on M, then f is a Clifford — Wolf translation on (M,d).*

$\lhd$ Let us take $x,y \in M$ and prove that $d(x,f(x)) = d(y,f(y))$. Since G acts transitively on M, there is $g \in G$ such that $g(x) = y$. Further, we have

$$d(y,f(y)) = d\big(g(x),f(g(x))\big) = d\big(g(x),g(f(x))\big) = d(x,f(x)),$$

since g is an isometry of (M,d) and $fg = gf$. $\rhd$

Definition 4. A metric space (M,d) is called *generalized normal homogeneous* (respectively, *Clifford — Wolf homogeneous*), if for every $x,y \in M$ there exists a $\delta(x)$-translation (respectively, *Clifford — Wolf translation*) of (M,d), moving x to y.

Clearly, any Clifford — Wolf translation is a $\delta(x)$-translation for all $x \in M$, any Clifford — Wolf homogeneous space is generalized normal homogeneous, and the latter one is homogeneous.

Since $\mathbb{R}^n$ is a commutative group (with respect to the vector addition), then every Euclidean space is Clifford — Wolf homogeneous by Proposition 1 (but it is easy to find all suitable CW-translations explicitly). Let us consider a more general example.

Example 1. Let G be a group, supplied with a metric d, that is bi-invariant, i. e. invariant both with respect to left and right shifts on G (for an element $a \in G$, the maps $l_a : x \mapsto a \cdot x$ and $r_a : x \mapsto x \cdot a$ are called the left shift and the right shift by a respectively). It is clear that the group of left shifts, as well as the group of right shifts, acts transitively on G. Moreover, every left shift centralizes the group of right shifts, hence, it is a Clifford — Wolf translation of (G, d). Analogously, every right shift is a Clifford — Wolf translation of (G, d). This implies that (G, d) is Clifford — Wolf homogeneous.

Note also that every odd-dimensional sphere (with the standard metric of constant curvature) is Clifford — Wolf homogeneous, see details e. g. in [4], [6, Chapter 7] or [9].

2. Finite Homogeneous Metric Spaces

Here we recall some important properties of finite homogeneous metric spaces. There are the following sources for such spaces (see details in [1]):

(1) a homogeneous space G/H of a finite group G by some its subgroup H, endowed with an invariant metric;

(2) a compact convex polytope in Euclidean space, whose isometry group acts transitively on the vertex set;

(3) a vertex-symmetric (vertex-transitive, in other terminology) connected finite graph with the natural metric;

(4) the Cayley graph of a finite group for a minimal generating set.

Definition 5. A map of metric spaces $f : (M_1, d_1) \to (M_2, d_2)$ is called a *submetry*, if it maps every closed ball $B(x, s) \subset (M_1, d_1)$ with center x and radius s onto the closed ball $B(f(x), s) \subset (M_2, d_2)$ with center $f(x)$ and radius s [10].

Definition 6. A finite homogeneous metric space (M, d) is called *normal homogeneous* if for its isometry group $\mathrm{Isom}\,(M, d)$ and its stabilizer H at a point $x_0 \in M$, there exists a subgroup Γ of the group $\mathrm{Isom}\,(M, d)$ which is transitive on M and a bi-invariant metric σ on Γ such that the canonical projection $\pi : (\Gamma, \sigma) \to (\Gamma/(\Gamma \cap H), d) = (M, d)$ is a submetry.

Remark 1. It should be noted that there are more restrictive definitions of *the normal homogeneity* for some special classes of metric spaces. For instance, that is the case with Riemannian manifolds, see details in [3, 5, 6].

The following result shows that the property to be generalized normal homogeneous (that is a pure metrical property) is equivalent to the property to be a normal homogeneous (that is an algebraic property in fact) in the case of finite metric spaces.

Proposition 2 [1]**.** *A finite metric space (M, d) is generalized normal homogeneous if and only if it is normal homogeneous.*

$\lhd$ *At first, let us prove that a finite normal homogeneous metric space (M, d) is generalized normal homogeneous.* Let us denote $\Gamma \cap H$ by H' (see Definition 6). We identify elements of M with left cosets $\alpha H' = \pi(\alpha)$, $\alpha \in \Gamma$. Since the canonical projection $\pi : (\Gamma, \sigma) \to (\Gamma/H', d) = (M, d)$ is a submetry, then the following statements hold: (I) the map π does not increase distances; (II) for every three points $x, y \in M$, $\xi \in \pi^{-1}(x)$, there exists a point $\eta \in \pi^{-1}(y)$ such that $\sigma(\xi, \eta) = d(x, y)$.

Let us consider some $x, y \in M$ and $\xi \in \pi^{-1}(x)$. We know (by (II)) that there is $\eta \in \pi^{-1}(y)$ such that $\sigma(\xi, \eta) = d(x, y)$. Let us consider $\gamma = \eta\xi^{-1}$. According to Example 1, the left shift by γ is a Clifford — Wolf translation of the space (Γ, σ). On the other hand, γ is an isometry of $(\Gamma/H' = M, d)$ with $\gamma(x) = y$ (since $\gamma\xi H' = \eta H'$). Further, for any $z \in M$ and any $\zeta \in \pi^{-1}(z)$, we have

$$d(x, \gamma(x)) = d(x, y) = \sigma(\xi, \eta) = \sigma(\xi, \gamma\xi) = \sigma(\zeta, \gamma\zeta)) \geqslant \quad \text{(by (I))}$$
$$\geqslant d(\pi(\zeta), \pi(\gamma\zeta)) = d(\zeta H', \gamma\zeta H') = d(z, \gamma(z)).$$

Therefore, γ is a $\delta(x)$-translation. Since $x, y \in M$ could be arbitrary, we get that the metric space (M, d) is generalized normal homogeneous.

Let us prove that a finite generalized normal homogeneous metric space (M, d) *is normal homogeneous.* We consider the group $G = \operatorname{Isom}(M, d)$ and the stabilizer H of a certain point $x_0 \in M$ in G. Let us define

$$\sigma(g, h) := \max_{x \in M} d(g(x), h(x)), \quad g, h \in G.$$

It is easy to verify that σ is a bi-invariant metric on G. We state that

$$\pi : (G, \sigma) \to (G/H, d) = (M, d)$$

is a submetry, hence, (M, d) is normal homogeneous. The definition of σ implies

$$d(\pi(g), \pi(h)) = d(g(x_0), h(x_0)) \leqslant \sigma(g, h)$$

for every $g, h \in G$, i. e. π does not increase the distance. Let x be any point in M. Since (M, d) is generalized normal homogeneous, there exists $g \in G$ such that $g(x_0) = x$ and $d(x_0, x) = d(x_0, g(x_0)) \geqslant d(y, g(y))$ for all $y \in M$ (i. e. g is a $\delta(x_0)$-translation). Therefore,

$$\sigma(e, g) = d(x_0, g(x_0)) = d(\pi(e), \pi(g)).$$

From the above reasoning it follows that $\pi(B(e, r)) = B(x_0, r) = B(\pi(e), r)$ for each number $r \geqslant 0$. Since the metric σ is left-invariant, we get $\pi(B(g, r)) = \pi(l_g(B(e, r))) = B(\pi(g), r)$ for any $r \geqslant 0$, $g \in G$, i. e. π is a submetry. ⊳

Let us denote by $FGBM, FGLM, FCWHS, FGNHS, FNHS, FHS$ respectively the classes of finite groups with bi-invariant metrics, finite groups with left-invariant metrics, finite Clifford — Wolf homogeneous spaces, finite generalized normal homogeneous spaces, finite normal homogeneous spaces, and finite homogeneous spaces. Proposition 2 implies the equality $FGNHS = FNHS$. It is known also that

$$FGBM \subset FCWHS \subset FGNHS = FNHS \subset FHS,$$
$$FGBM \subset FGLM \subset FHS.$$

Moreover, all the above inclusions are strict, see details in [1]. In what follows we will consider mostly two subclasses of the class of finite homogeneous metric spaces: finite Clifford — Wolf homogeneous metric spaces and finite (generalized) normal homogeneous spaces.

3. Finite Homogeneous Subspace of Euclidean Spaces

In this section we deal with finite subsets of Euclidean space $\mathbb{R}^n$. We assume that any such set M is supplied with the metric d induced from $\mathbb{R}^n$.

Since the barycenter of a finite system of material points (with one and the same mass) in any Euclidean space is preserved for any bijection (in particular, any isometry) of this system, we immediately get the following result.

Proposition 3 [1]**.** *Let $M = \{x_1, \dots, x_m\}$, $m \geqslant n+1$, be a finite homogeneous metric subspace of Euclidean space $\mathbb{R}^n$, $n \geqslant 2$, which does not lie in a hyperplane. Then M is the vertex set of a convex polytope P, that is situated in some sphere in $\mathbb{R}^n$ with radius $r > 0$ and center $x_0 = \frac{1}{m} \cdot \sum_{k=1}^{m} x_k$. In particular, $\operatorname{Isom}(M, d) \subset O(n)$. Up to a similarity, any homogeneous finite metric subspace in $\mathbb{R}^n$, $n \geqslant 2$, is a homogeneous metric subspace of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.*

This result shows that *the theory of convex polytopes* is very important for the study of finite homogeneous subspaces of Euclidean spaces. Now, we recall several important definitions. For a more detailed acquaintance with the theory of convex polytopes, we recommend [11–15].

We say that a n-dimensional polytope P in $\mathbb{R}^n$ is *homogeneous* (or *vertex-transitive*) if its isometry group acts transitively on the set of its vertices. Further, P is called a *polytope with regular faces* (respectively, a *polytope with congruent faces*), if all its facets are regular (respectively, congruent) polytopes.

A one-dimensional polytope is a closed segment, bounded by two endpoints. It is regular by definition. Two-dimensional regular polytopes are regular polygons on Euclidean plane. For other dimensions, regular polytopes are defined inductively. A convex n-dimensional polytope for $n \geqslant 3$ is called *regular*, if it is homogeneous and all its facets are regular polytopes congruent to each other. This definition is equivalent to other definitions of regular convex polytopes (see [16]).

We also recall the definition of semiregular convex polytopes. For $n = 1$ and $n = 2$, semiregular polytopes are defined as regular. A convex n-dimensional polytope for $n \geqslant 3$ is called *semiregular* if it is homogeneous and all its facets are regular polytopes.

A generalization of the class of semiregular polytopes is the class of uniform polytopes. For $n \leqslant 2$, uniform polytopes are defined as regular. For other dimensions, uniform polytopes are defined inductively. A convex n-dimensional polytope for $n \geqslant 3$ is called *uniform* if it is homogeneous and all its facets are uniform polytopes. In particular, for $n = 3$, the classes of uniform and semiregular polytopes coincide, and for $n = 4$ the facets of the uniform polytope must be semiregular three-dimensional polytopes. This class of polyhedra is far from complete classification, see known results in [16, 17].

The classification of regular polytopes of arbitrary dimension was first obtained by Ludwig Schläfli and is presented in his book [18], see also Harold Coxeter's book [12]. The list of semiregular polytopes of arbitrary dimension was first presented without proof in Thorold Gosset's paper [19]. Later this list appeared in the work of Emanuel Lodewijk Elte [20]. The proof of the completeness of this list was obtained much later by Gerd Blind and Rosvita Blind, see [21] and the references therein. Semiregular (non-regular) polytopes in $\mathbb{R}^n$ for $n \geqslant 4$ are called *Gosset polytopes*. A lot of additional information can be found in [22].

4. Regular and Semiregular Polytopes

We briefly recall the classification of regular and semiregular polytopes in Euclidean spaces. Each regular n-dimensional polytope is characterized by its Schläfli symbol $\{p_1, p_2, \dots, p_{n-1}\}$, an ordered set of $(n-1)$ natural numbers. A *vertex figure* of n-dimensional regular polytope, $n \geqslant 3$, is a $(n-1)$-dimensional polytope, which is the convex hull of the vertices, having a common edge with a given vertex and different from it. Faces of dimension $n-1$ (hyperfaces) of a n-dimensional polytope commonly referred to as *facets*. Note that they are also called *cells* for $n = 4$.

Table 1. Regular 3-dimensional polyhedra

Polyhedron	V	E	F	α	Face	Schläfli symbol
Tetrahedron	4	6	4	$2\arcsin(1/\sqrt{3})$	$\triangle$	$\{3,3\}$
Cube (hexahedron)	8	12	6	$\pi/2$	$\square$	$\{4,3\}$
Octahedron	6	12	8	$2\arcsin(\sqrt{2/3})$	$\triangle$	$\{3,4\}$
Dodecahedron	20	30	12	$2\arcsin\left(\sqrt{\varphi}/\sqrt[4]{5}\right)$	⬠	$\{5,3\}$
Icosahedron	12	30	20	$2\arcsin(\varphi/\sqrt{3})$	$\triangle$	$\{3,5\}$

An one-dimensional polytope (closed segment) is regular and is represented by the Schläfli symbol $\{\,\}$. Two-dimensional regular polyhedra (polygons) have equal sides and are inscribed in a circle. A regular p-gon is represented by the Schläfli symbol $\{p\}$.

In dimensions $n \geqslant 3$, the Schläfli symbol can be defined inductively: for a n-dimensional polytope M it is equal to $\{p_1, p_2, \dots, p_{n-1}\}$, where p_1 is the number of sides of an (arbitrary) two-dimensional face of the polytope M, and $\{p_2, \dots, p_{n-1}\}$ is the Schläfli symbol for the vertex figure of the polytope M. It is clear that the facet of the polytope M has Schläfli symbol $\{p_1, p_2, \dots, p_{n-2}\}$.

Table 2. Regular 4-dimensional polytopes

Polytope	V	E	F	C	Cell	Schläfli symbol
Hypertetrahedron or 5-cell	5	10	10	5	tetrahedron	$\{3,3,3\}$
Hypercube or 8-cell	16	32	24	8	cube	$\{4,3,3\}$
Hyperoctahedron or 16-cell	8	24	32	16	tetrahedron	$\{3,3,4\}$
24-cell	24	96	96	24	octahedron	$\{3,4,3\}$
120-cell	600	1200	720	120	dodecahedron	$\{5,3,3\}$
600-cell	120	720	1200	600	tetrahedron	$\{3,3,5\}$

In three-dimensional space, a regular polyhedron with the Schläfli symbol $\{m,n\}$ has regular faces of the type $\{m\}$ and a regular vertex figure with the symbol $\{n\}$. For regular three-dimensional polyhedra, the vertex figure is a polygon. It is well known that there are only five regular three-dimensional polyhedra: the tetrahedron, cube, octahedron, dodecahedron and icosahedron with the Schläfli symbols $\{3,3\}$, $\{4,3\}$, $\{3,4\}$, $\{5,3\}$ and $\{3,5\}$ respectively. These polyhedra are traditionally called *Platonic solids*. Some important properties of these polyhedra could be found in Table 1, where V, E, and F mean respectively the numbers of vertices, edges, and faces; α is the dihedral angle; the number $\varphi := \frac{1+\sqrt{5}}{2}$ is known as *the golden ratio*.

A regular 4-dimensional polytope with the Schläfli symbol $\{m,n,s\}$ has cells of the type $\{m,n\}$, 2-faces of the type $\{m\}$, and the vertex figures $\{n,s\}$. The list of 4-dimensional regular polytopes together with their important characteristics are given in Table 2, where V, E, F, and C mean respectively the numbers of vertices, edges, faces, and cells (3-dimensional faces). A more detailed description of the structure of four-dimensional regular polytopes could be found e. g. in [7, Section 3].

For each dimension $n \geqslant 5$, there exists three regular polytopes: the n-dimensional simplex, the hypercube (n-cube) and the hyperoctahedron (n-orthoplex). Important characteristics of these polytopes are given in Table 3.

We now proceed to a brief description of the semiregular (non-regular) polytopes.

In three-dimensional space (in addition to Platonic solids), there are the following semiregular polyhedra: 13 *Archimedean solids* and two infinite series of regular prisms and right antiprisms.

A *right prism* is a polyhedron whose two faces (called bases) are congruent (equal) polygons, lying in parallel planes, while other faces (called lateral ones) are rectangles (perpendicular to the bases). It is easy to see that the vertex set of every right prism is Clifford — Wolf homogeneous. If lateral faces are squares then the prism is said to be *regular*. In this case we get an infinite family of semiregular convex polyhedra.

A *right antiprism* is a semiregular polyhedron, whose two parallel faces (bases) are equal regular n-gons, while other $2n$ (lateral) faces are regular triangles. Note that the octahedron is an antiprism with triangular bases. It is easy to check that the vertex set of every right prism is Clifford — Wolf homogeneous.

A detailed description of Archimedean solids could be found in Sections 4 and 5 of [7].

Table 3. Regular n-dimensional polytopes for $n \geqslant 5$

Polytope	Schläfli symbol	Number of k-faces	Facet	Vertex figure
n-simplex	$\{3,3,\dots,3,3\}$	C_{n+1}^{k+1}	$\{3,3,\dots,3\}$	$\{3,\dots,3,3\}$
n-cube	$\{4,3,\dots,3,3\}$	$2^{n-k}C_n^k$	$\{4,3,\dots,3\}$	$\{3,\dots,3,3\}$
n-orthoplex	$\{3,3,\dots,3,4\}$	$2^{k+1}C_n^{k+1}$	$\{3,3,\dots,3\}$	$\{3,\dots,3,4\}$

According to the classification of semiregular polytopes in $\mathbb{R}^n$, $n \geqslant 4$ (see [19] and [21]), besides regular polytopes, there are three semiregular polytopes in $\mathbb{R}^4$ and one semiregular polytope in $\mathbb{R}^n$ for $n = 5, 6, 7, 8$.

Recall that *the rectified polytope* P is the convex hull of the midpoints of the edges of P. Note also that *rectification*, also known as *critical truncation* or *complete truncation* is the process of truncating a polytope by marking the midpoints of all its edges, and cutting off its vertices at those points.

For $n = 4$ we have exactly three semiregular polytopes: the rectified 4-simplex, rectified 600-cell, and snub 24-cell. A detailed description of these polytopes could be found in Section 5 of [8].

The unique (up to similarity) semiregular Gosset polytope in $\mathbb{R}^n$ for $n \in \{5, 6, 7, 8\}$ we denote by the symbol Goss_n. Detailed descriptions of these polytopes could be found in Sections 6, 7, 8, and 9 of [8] respectively.

Example 2. Let us consider a brief explicit description of Goss_6. This polytope can be implemented in different ways. Let us set it with the coordinates of the vertices in $\mathbb{R}^6$, as it is done in [20]. Let us put $a = \frac{\sqrt{2}}{4}$ and $b = \frac{\sqrt{6}}{12}$. We define the points $A_i \in \mathbb{R}^6$, $i = 1, \dots, 27$, as follows:

$A_1 = (0,0,0,0,0,4b)$, $A_2 = (a,a,a,a,a,b)$, $A_3 = (-a,-a,a,a,a,b)$,
$A_4 = (-a,a,-a,a,a,b)$, $A_5 = (-a,a,a,-a,a,b)$, $A_6 = (-a,a,a,a,-a,b)$,
$A_7 = (a,-a,-a,a,a,b)$, $A_8 = (a,-a,a,-a,a,b)$, $A_9 = (a,-a,a,a,-a,b)$,
$A_{10} = (a,a,-a,-a,a,b)$, $A_{11} = (a,a,-a,a,-a,b)$, $A_{12} = (a,a,a,-a,-a,b)$,
$A_{13} = (-a,-a,-a,-a,a,b)$, $A_{14} = (-a,-a,-a,a,-a,b)$, $A_{15} = (-a,-a,a,-a,-a,b)$,
$A_{16} = (-a,a,-a,-a,-a,b)$, $A_{17} = (a,-a,-a,-a,-a,b)$, $A_{18} = (2a,0,0,0,0,-2b)$,
$A_{19} = (0,2a,0,0,0,-2b)$, $A_{20} = (0,0,2a,0,0,-2b)$, $A_{21} = (0,0,0,2a,0,-2b)$,
$A_{22} = (0,0,0,0,2a,-2b)$, $A_{23} = (-2a,0,0,0,0,-2b)$, $A_{24} = (0,-2a,0,0,0,-2b)$,
$A_{25} = (0,0,-2a,0,0,-2b)$, $A_{26} = (0,0,0,-2a,0,-2b)$, $A_{27} = (0,0,0,0,-2a,-2b)$.

The Gosset polytope Goss_6 is the convex hull of these points. It is easy to check that $d(A_1, A_i) = 1$ for $2 \leqslant i \leqslant 17$ and $d(A_1, A_i) = \sqrt{2}$ for $18 \leqslant i \leqslant 27$.

It is clear that the points $A_2 - A_{17}$ are vertices of a five-dimensional semi-hypercube (the corresponding hypercube has 32 vertices of the form $(\pm a, \pm a, \pm a, \pm a, \pm a, b)$), and the points $A_{18} - A_{27}$ are the vertices of the five-dimensional hyperoctahedron (orthoplex), which is a facet of the polytope Goss_6 (lying in the hyperplane $x_6 = -2b$). The origin $O = (0, 0, 0, 0, 0, 0) \in \mathbb{R}^6$ is the center of the hypersphere described around Goss_6 with radius $4b = \sqrt{2/3}$.

5. Main Results

In [1, 7, 8], the authors obtained the complete description of the metric properties of the sets of vertices of regular and semiregular polytopes in Euclidean spaces from the point of view of the normal homogeneity and the Clifford — Wolf homogeneity. Recall that any set M in the Euclidean space $\mathbb{R}^n$ is supposed to be supplied with the metric d induced from $\mathbb{R}^n$. Here we collect all related results in the following theorem.

Table 4. Metric properties of regular and semiregular polytopes

№	Polytope	Dimension	Regularity	(NH, CWH)	Source
1	n-simplex	n	R	$(+,+)$	[1]
2	n-cube	n	R	$(+,+)$	[1]
3	n-orthoplex	n	R	$(+,+)$	[1]
4	any regular polygon	2	R	$(+,+)$	[1]
5	dodecahedron	3	R	$(-,-)$	[1]
6	icosahedron	3	R	$(+,-)$	[1]
7	24-cell	4	R	$(+,+)$	[7]
8	120-cell	4	R	$(-,-)$	[7]
7	600-cell	4	R	$(+,+)$	[7]
8	any regular prism	3	SR	$(+,+)$	[7]
9	any right antiprism	3	SR	$(+,+)$	[7]
10	any Archimedean solid	3	SR	$(-,-)$	[7]
11	rectified 4-simplex	4	SR	$(+,-)$	[8]
12	snub 24-cell	4	SR	$(-,-)$	[8]
13	rectified 600-cell	4	SR	$(-,-)$	[8]
14	Goss_5	5	SR	$(+,+)$	[8]
15	Goss_6	6	SR	$(+,-)$	[8]
16	Goss_7	7	SR	$(+,-)$	[8]
17	Goss_8	8	SR	$(+,+)$	[8]

Theorem 1 [1, 7, 8]**.** *For a given regular or semiregular polytope P in $\mathbb{R}^n$, the vertex set M of P is normal homogeneous or Clifford — Wolf homogeneous if and only if there is the sign "+" in the suitable place of the intersection of the row corresponding to P with the fifth column of Table 4, where NH means the normal homogeneity and CWH means the Clifford — Wolf homogeneity.*

Remark 2. In the fourth column of Table 4 we clarify the degree of regularity of P: R and SR mean respectively a regular polytope and a semiregular (non-regular) polytope. The last column of Table 4 contains the sources for the corresponding results.

We discuss in more detail some tools that were used for obtaining of suitable results.

The most usual way to prove, that a given metric space (M, d) is Clifford — Wolf homogeneous, is to supply M with a group structure, such that d is invariant both under the left and right shifts, see Example 1. For instance, the vertex set M of a regular polygon with m vertices could be identified with the cyclic group C_m (with the generator that is a rotation around center of the polygon with rotation angle $2\pi/m$). Hence the vertices of every regular two-dimensional polyhedron (regular polygon) form Clifford — Wolf homogeneous metric spaces. The vertex sets of the n-dimensional simplex and the hypercube (n-cube) can be considered respectively as the cyclic group C_{n+1} and the group $(\mathbb{Z}_2)^n$ with bi-invariant metrics, see [1, Corollary 2] (the same idea works in the case of the n-dimensional semi-hypercube, see [8, Proposition 19]). For $n = 1$ we get the case of one-dimensional polytope (segment) in this construction. Note that the vertex sets of the 24-cell, dysphenoidal 288-cell, and 600-cell could be identified with some subgroups of the group S^3 (the group of unit quaternions), hence, these sets are also Clifford — Wolf homogeneous metric spaces, see [7, Proposition 3]. We specially note that *the dysphenoidal* 288-*cell* is neither regular, nor semiregular, nor even uniform, see Section 3 in [7].

It should be noted that we used quite special methods for some of Clifford — Wolf homogeneous metric spaces. For instance, the hyperoctahedron (n-ortoplex) is Clifford — Wolf homogeneous metric spaces in $\mathbb{R}^n$ for any $n \geqslant 1$ by [1, Corollary 4].

Let us consider one idea how to prove that a vertex set of a given polytope is not normal homogeneous. It is based on the following

Proposition 4. *Let M be the vertex set of a polytope $P \subset \mathbb{R}^n$. Suppose that there are adjacent each to other points $O, O' \in M$ such that*

1) *for any vertex $Q' \in M$ adjacent to O' and $Q' \neq O$ we get $\angle OO'Q' \geqslant \frac{\pi}{2}$;*

2) *there are two distinct vertices $Q_1, Q_2 \in M$ adjacent to O and distinct from O' such that $\angle Q_iOO' > \frac{\pi}{2}$, $i = 1, 2$.*

Then (M, d) is not normal homogeneous.

$\lhd$ Denote $d(O, O')$ by ρ. Suppose that (M, d) is normal homogeneous. Then there is an isometry ψ of the metric space (M, d), shifting all points by a distance at most ρ and such that $\psi(O) = O'$ (ψ is a δ-shift at the point O).

Since ψ is an isometry and the vertex Q_i is adjacent to O, then $\psi(Q_i)$ is adjacent to O', $i = 1, 2$. Since $\psi(Q_2) \neq \psi(Q_1)$, one of these point, say $\psi(Q_1)$, is distinct from O. Then we have $\angle OO'\psi(Q_1) \geqslant \pi/2$ and $\angle Q_1OO' > \pi/2$, therefore, $d(Q_1, \psi(Q_1)) > \rho$ that impossible (even the orthogonal projection of the line segment $[Q_1, \psi(Q_1)]$ to the straight line OO' is longer than ρ). Hence, the map ψ with desirable properties does not exist. The proposition is proved. $\rhd$

This proposition could be used to prove that the vertex sets of the dodecahedron in $\mathbb{R}^3$, the 120-cell in $\mathbb{R}^4$, and Archimedian solids are not normal homogeneous.

To study all other regular and semiregular polytopes, we refer the reader to detailed reasoning in [1], [7], and [8].

References

1. *Berestovskii, V. N. and Nikonorov, Yu. G.* Finite Homogeneous Metric Spaces, *Siberian Mathematical Journal*, 2019, vol. 60, no. 5, pp. 757–773. DOI: 10.1134/S0037446619050021.
2. *Berestovskii, V. N and Nikonorov, Yu. G.* Killing Vector Fields of Constant Length on Riemannian Manifolds, *Siberian Mathematical Journal*, 2008, vol. 49, no. 3, pp. 395–407. DOI: 10.1007/s11202-008-0039-3.
3. *Berestovskii, V. N and Nikonorov, Yu. G.* On δ-Homogeneous Riemannian Manifolds, *Differential Geometry and its Applications*, 2008, vol. 26, no. 5, pp. 514–535. DOI: 10.1016/j.difgeo.2008.04.003.
4. *Berestovskii, V. N. and Nikonorov, Yu. G.* Clifford — Wolf Homogeneous Riemannian Manifolds, *Journal of Differential Geometry*, 2009, vol. 82, no. 3, pp. 467–500. DOI: 10.4310/jdg/1251122544.

5. *Berestovskii, V. N. and Nikonorov, Yu. G.* Generalized Normal Homogeneous Riemannian Metrics on Spheres and Projective Spaces, *Annals of Global Analysis and Geometry*, 2014, vol. 45, no. 3, pp. 167–196. DOI: 10.1007/s10455-013-9393-x.
6. *Berestovskii, V. N. and Nikonorov, Yu. G. Riemannian Manifolds and Homogeneous Geodesics*, Springer Monographs in Mathematics, Cham, Springer, 2020. DOI: 10.1007/978-3-030-56658-6.
7. *Berestovskii, V. N. and Nikonorov, Yu. G.* Finite Homogeneous Subspaces of Euclidean Spaces, *Siberian Advances in Mathematics*, 2021, vol. 31, no. 3, pp. 155–176. DOI: 10.1134/S1055134421030019.
8. *Berestovskii, V. N. and Nikonorov, Yu. G.* Semiregular Gosset polytopes, *Izvestiya: Mathematics*, 2022, vol. 86, no. 4 (accepted). DOI:10.1070/IM9169.
9. *Wolf, J. A. Spaces of Constant Curvature*, 6th ed., AMS Chelsea Publishing, Providence, RI, 2011. DOI: 10.1090/chel/372.
10. *Berestovskii, V. N. and Guijarro, L.* A Metric Characterization of Riemannian Submersions, *Annals of Global Analysis and Geometry*, 2000, vol. 18, no. 6, pp. 577–588. DOI: 10.1023/A:1006683922481
11. *Berger, M. Geometry I.* Universitext, Springer-Verlag, Berlin, 2009.
12. *Coxeter, H. S. M. Regular Polytopes*, 3d ed., New York, Dover, 1973.
13. *Cromwell, P. R. Polyhedra*, Cambridge, Cambridge Univ. Press, 1997.
14. *Grünbaum, B. Convex Polytopes*, 2nd ed., Graduate Texts in Mathematics, 221, New York, Springer, 2003. DOI: 10.1007/978-1-4613-0019-9.
15. *Smirnov, E. Yu. Reflection Groups and Regular Polyhedra*, 2nd ed., MCCME, Moscow, 2018, 56 p. (in Russian).
16. *Martini, H.* A Hierarchical Classification of Euclidean Polytopes with Regularity Properties, *Polytopes: Abstract, Convex and Computational*, Proc. of the NATO Advanced Study Institute, Scarborough, Ontario, Canada, August 20–September 3, 1993, Eds. T. Bisztriczky et al. Dordrecht: Kluwer Academic Publishers. NATO ASI Ser., Ser. C, Math. Phys. Sci. 440, 1994, pp. 71–96. DOI: 10.1007/978-94-011-0924-6_4.
17. Four-Dimensional Euclidean Space. URL: http://eusebeia.dyndns.org/4d/index.
18. *Schläfli, L. Theorie der Vielfachen Kontinuität.* Hrsg. im Auftrage der Denkschriften-Kommission der Schweizerischen Naturforschenden Gesellschaft von J.H. Graf. Zürich, Basel, Georg & Co, 1901.
19. *Gosset, Th.* On the Regular and Semi-Regular Figures in Space of n Dimensions, *Messenger Math.*, 1900, vol. 29, pp. 43–48.
20. *Elte, E. L. The Semiregular Polytopes of the Hyperspaces*, 1912, Groningen, University of Groningen, URL: https://quod.lib.umich.edu/u/umhistmath/ABR2632.0001.001, 20.03.2021.
21. *Blind, G. and Blind, R.* The Semiregular Polytopes, *Comment. Math. Helv.*, 1991, vol. 66, no. 1, pp. 150–154. DOI: 10.1007/BF02566640.
22. Dutour Sikirić, M. URL: http://mathieudutour.altervista.org/Regular/, 20.03.2021.

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О КОНЕЧНЫХ ОДНОРОДНЫХ МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ

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Аннотация. Работа представляет собой обзор недавно полученных результатов о конечных однородных метрических пространствах. Основным предметом обсуждения является классификация правильных и полуправильных многогранников в евклидовых пространствах по наличию у множеств их вершин свойств нормальной однородности или однородности по Клиффорду — Вольфу. Каждое конечное однородное метрическое подпространство евклидова пространства представляет собой множество вершин компактного выпуклого многогранника с группой изометрий, транзитивной на множестве вершин, причем все эти вершины лежат на некоторой сфере. Таким образом, изучение таких подмножеств тесно связано с теорией выпуклых многогранников в евклидовых пространствах. Нормальная обобщенная однородность и однородность по Клиффорду — Вольфу описывают более сильные свойства, чем однородность. Поэтому естественно сначала проверить наличие этих свойств для вершинных множеств правильных и полуправильных многогранников. Помимо классификационных результатов, статья содержит описание основных инструментов для исследования соответствующих объектов.

Ключевые слова: архимедово тело, конечное нормальное однородное метрическое пространство, конечное однородное метрическое пространство, конечное однородное по Клиффорду — Вольфу метрическое пространство, многогранник Госсета, платоново тело, полуправильный многогранник, правильный многогранник.

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