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EXTENSION OF AN ALMOST f-ALGEBRA MULTIPLICATION

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It is proved that an almost f-algebra multiplication and a d-algebra multiplication defined on a majorizing vector sublattice of a Dedekind complete vector lattice can be extended to the whole vector lattice by using purely algebraic and order theoretical means.

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As well-known, the multiplication in an Archimedean f-algebra A can be extended (uniquely) to multiplication in A^δ, the Dedekind completion of A, in such a manner that A^δ is again an f-algebra with respect to this extended multiplication. The corresponding results for Archimedean lattice-ordered algebras, almost f-algebras and d-algebras were conjectured by Huijsmans in his survey paper [7]. We give a short historical account. This conjecture was proved by Buskes and van Rooij in [5, Theorem 4.1] for the class of almost f-algebras. In contrast to the original approach of Buskes and van Rooij, Boulabiar and the author gave in [4] an order theoretical and algebraic approach of the Buskes and van Rooij theorem. As an application, Boulabiar and the author gave in [4] an affirmative answer to Huijsmans's question for the class of commutative d-algebras. Some years after in [6], the author proved that the multiplication in an Archimedean (not necessarily commutative) d-algebra A can be extended to a multiplication in the Dedekind completion A^δ of A in such a fashion that A^δ becomes a d-algebra under this extended multiplication. Therefore, it seems natural to ask can an almost f-algebra (respectively, a d-algebra) multiplication given on a majorizing vector sublattice of a Dedekind complete vector lattice be extended to an almost f-algebra (respectively, a d-algebra) multiplication on the ambient vector lattice. In [8] Kusraev gave a positive answer to this question for the class of almost f-algebras. However, their proof makes use of representation theory which we wish to avoid in this work. The principal purpose of this paper is to give another proof to Kusraev's result (see Theorem 1 below) using purely algebraic and order theoretical means.

We assume that the reader is familiar with the notion of vector lattices (or Riesz spaces).

It is convenient to use the monograph [1] for basic information concerning the general theory of vector lattices. More about almost f-algebras and d-algebras can be found in [2].

Our approach relies heavily on a theorem due to Boulabiar and the author about the structure of almost f-algebras. The theorem in question is that if $(A,*)$ is an almost f-algebra if and only if there exist (i) a universally complete f-algebra B with unit element, multiplication of which is denoted by juxtaposition; (ii) a lattice homomorphism σ from A into B such that B is the universal completion of $\sigma(A)$; (iii) a positive operator T from $\sigma(A)_p = \{\sigma(a)\sigma(b) : a, b \in A\}$ into A; and (iv) a positive element ξ of B, such that $a * b = T(\sigma(a)\sigma(b))$, $\sigma \circ T(u) = \xi u$ and $(a * b) * c = T\big(\xi\sigma(a)\,\sigma(b)\,\sigma(c)\big)$ hold for all $a, b, c \in A$ and $u \in \sigma(A)_p$ (see [4]).

Theorem 1. *Let A be an almost f-algebra and a majorizing vector sublattice of a Dedekind complete vector lattice E. Then the multiplication in A extends to an almost f-algebra multiplication in E.*

$\lhd$ Consider B, σ, T, and ξ as defined in the Boulabiar and author's theorem previously mentioned. In view of [1, Theorem 7.17], σ extends to a lattice homomorphism $\widetilde{\sigma}$ from E into $\sigma(A)^{\delta}$. Moreover, from the fact that A is majorizing in E, it follows that $\sigma(A)$ is majorizing in $\widetilde{\sigma}(E)$. Therefore, $\sigma(A)_p$ is majorizing in $\widetilde{\sigma}(E)_p$. According to the Kantorovič theorem [1, Theorem 2.8], T extends to a positive operator $\widetilde{T}$ from $\widetilde{\sigma}(E)_p$ into E. Define now a multiplication $*$ in E by putting

$$a * b = \widetilde{T}(\widetilde{\sigma}(a)\,\widetilde{\sigma}(b)) \quad \text{for all} \quad a, b \in E.$$

Of course, this multiplication extends that one in A and it is easy to see that if $a \wedge b = 0$ then $a*b = 0$ for all $a, b \in E$. This implies that associativity of $*$ is sufficient to let E be an almost f-algebra with respect to the extended multiplication $*$. We claim that $\widetilde{\sigma} \circ \widetilde{T}(u) = \xi u$ for all $u \in \widetilde{\sigma}(E)_p$. A moment's reflection shows that $\widetilde{\sigma}(E)$ is majorizing vector sublattice of $\sigma(A)^{\delta}$. Now from the fact that $\sigma(A)_p$ is majorizing and order dense sublattice of $\sigma(A)_p^{\delta}$ it follows that $\widetilde{\sigma}(E)_p$ is majorizing and order dense sublattice of $\sigma(A)_p^{\delta}$ (because $\sigma(A)_p \subset \widetilde{\sigma}(E)_p \subset \sigma(A)_p^{\delta}$). Consequently,

$$\sup\{\sigma \circ T(v) = \xi v : 0 \leqslant v \leqslant u\} \leqslant \widetilde{\sigma} \circ \widetilde{T}(u);$$
$$\inf\{\sigma \circ T(v) = \xi v : 0 \leqslant v \leqslant u\} \text{ for all } u \in \widetilde{\sigma}(E)_p.$$

Now, by using order continuity of f-algebra multiplication, we obtain:

$$\widetilde{\sigma} \circ \widetilde{T}(u) = \xi\ u \text{ for all } \ u \in \widetilde{\sigma}(E)_p.$$

Finally, let $a, b, c \in E$,

$$(a*b)*c = \widetilde{T}(\widetilde{\sigma}(a*b)\widetilde{\sigma}(c)) = \widetilde{T}(\widetilde{\sigma} \circ \widetilde{T}(\widetilde{\sigma}(a)\widetilde{\sigma}(b))\widetilde{\sigma}(c)) = \widetilde{T}(\xi\widetilde{\sigma}(a)\widetilde{\sigma}(b))\widetilde{\sigma}(c)$$

and the theorem follows. $\rhd$

So far we are unable to prove or disprove the corresponding result for the class of d-algebras, the main reason being that a d-algebra needs not be commutative. This means that the method used for almost f-algebras does not work in the case of d-algebras. Nevertheless, if we impose this additional condition then the situation improves as it is shown in the following theorem.

Theorem 2. *Let A be a commutative d-algebra and a majorizing vector sublattice of a Dedekind complete vector lattice E. Then the multiplication in A extends to a commutative d-algebra multiplication in E.*

$\lhd$ Since any commutative d-algebra is an almost f-algebra, we obtain: σ, ξ, and T a lattice homomorphism, such that $a*b = T(\sigma(a)\sigma(b))$, $\sigma \circ T(u) = \xi u$ and $(a*b)*c = T\big(\xi\sigma(a)\sigma(b)\sigma(c)\big)$ hold for all $a, b, c \in A$ and $u \in \sigma(A)_p$ (for details on this see [4, Corollary 3]). Now, we deduce from the proof of preceding theorem that σ extends to a lattice homomorphism $\widetilde{\sigma}$ from E into $\sigma(A)^{\delta}$ and T extends to a positive operator $\widetilde{T}$ from $\widetilde{\sigma}(E)_p$ into E such that

$$a*b = \widetilde{T}(\widetilde{\sigma}(a)\widetilde{\sigma}(b)), \quad (a*b)*c = \widetilde{T}(\xi\widetilde{\sigma}(a)\widetilde{\sigma}(b))\widetilde{\sigma}(c)$$

hold for all $a, b, c \in E$. On the other hand, T is a lattice homomorphism. Therefore, according to [1, Theorem 7.17], $\widetilde{T}$ can be chosen to be a lattice homomorphism. Which that we wanted. $\rhd$

At the end we note that the extension obtained in Theorem 1 and Theorem 2 is far from being unique and this is illustrated by the following example (see [3]).

Example 3. Let E be the Archimedean vector lattice of all bounded real valued continuous functions on $\mathbb{N}$. With the pointwise addition, scalar multiplication, product, and ordering, E is an algebra and a Dedekind complete vector lattice. Furthermore, if $\beta\mathbb{N}$ denotes the Stone–Čech compactification of $\mathbb{N}$, then any function f in E extends uniquely to a function f^{β} in the vector lattice of all real valued continuous functions on the compact Hausdorff space $\beta\mathbb{N}$. Now let A be the vector sublattice of E of all f for which there exist real numbers ℓ_f and x_f such that the inequality $|f - \ell_f 1| \leqslant x_f h$ holds in E where 1 and g are the elements of E defined by

$$1(n) = 1 \text{ and } h(n) = \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

A moment's reflection shows that A is a majorizing vector sublattice of the Dedekind complete vector lattice E. Now we define a multiplication $*$ in A by

$$f * g = \ell_f \ell_g 1 \text{ for all } f, g \in A.$$

Clearly, $(A, *)$ is a commutative d-algebra. For any $p \in \beta\mathbb{N}\backslash\mathbb{N}$ the multiplication $*$ of A extends to a d-algebra multiplication $*$ on E by

$$f * g = f^{\beta}(p) g^{\beta}(p) 1 \text{ for all } f, g \in E.$$

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ПРОДОЛЖЕНИЕ УМНОЖЕНИЯ ПОЧТИ f-АЛГЕБРЫ

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Установлено, что чисто алгебраическими и порядковыми средствами можно доказать утверждение: умножения почти f-алгебры и d-алгебры, определенные на мажорирующей подрешетке дедекиндово полной векторной решетки, можно продолжить на всю векторную решетку.

Ключевые слова: почти f-алгебра, d-алгебра, f-алгебра, векторная решетка.